Legendre Polynomials from Gram−Schmidt Orthogonalisation in $L^2[-1, 1]$

```
e0 = 1;
e1 = x;
e2 = x^2;
e3 = x^3;
e4 = x^4;
e5 = x^5;
e6 = x^6;
e7 = x^7;
e8 = x^8;
e9 = x^9;
```

$$\text{Leng}[a\_, b\_] := \int_{-1}^{+1} a\, b\, dx$$

$$\text{length}[a\_] := \int_{-1}^{+1} a\, a\, dx$$

```
General::spell1 :
 Possible spelling error: new symbol name "length" is similar to existing symbol "Length". More…
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The f_n are the normalised basis after applying the Gram−Schmidt Orthogonalization on the e_n. One has $f_n(x) = \sqrt{\frac{2}{2n+1}}\, P_n(x)$, where $P_n(x)$ is the Legendre polynomial of order n.

$$f_0 = \sqrt{1/\text{Leng}[e_0, e_0]}\; e_0$$

$$f_1 = \sqrt{1/\text{Leng}[e_1, e_1]}\; e_1$$

$$f_2 = \text{Simplify}\left[\frac{e_2 - \text{Leng}[f_0, e_2]\, f_0}{\sqrt{\text{length}[e_2 - \text{Leng}[f_0, e_2]\, f_0]}}\right]$$

$$f_3 = \text{Simplify}\left[\frac{e_3 - \text{Leng}[f_1, e_3]\, f_1}{\sqrt{\text{length}[e_3 - \text{Leng}[f_1, e_3]\, f_1]}}\right]$$

$$f_4 = \text{Simplify}\left[\frac{e_4 - \text{Leng}[f_0, e_4]\, f_0 - \text{Leng}[f_2, e_4]\, f_2}{\sqrt{\text{length}[e_4 - \text{Leng}[f_0, e_4]\, f_0 - \text{Leng}[f_2, e_4]\, f_2]}}\right]$$

$$f_5 = \text{Simplify}\left[\frac{e_5 - \text{Leng}[f_1, e_5]\, f_1 - \text{Leng}[f_3, e_5]\, f_3}{\sqrt{\text{length}[e_5 - \text{Leng}[f_1, e_5]\, f_1 - \text{Leng}[f_3, e_5]\, f_3]}}\right]$$

$$\text{temp} = \text{Simplify}\left[e_6 - \sum_{j=0}^{2} \text{Leng}[f_{2j}, e_6]\, f_{2j}\right];$$

$$f_6 = \text{Simplify}\left[\frac{\text{temp}}{\sqrt{\text{length}[\text{temp}]}}\right]$$

$$\text{temp} = \text{Simplify}\left[e_7 - \sum_{j=0}^{2} \text{Leng}[f_{2j+1}, e_7]\, f_{2j+1}\right];$$

$$f_7 = \text{Simplify}\left[\frac{\text{temp}}{\sqrt{\text{length}[\text{temp}]}}\right]$$

$$\text{temp} = \text{Simplify}\left[e_8 - \sum_{j=0}^{3} \text{Leng}[f_{2j}, e_8]\, f_{2j}\right];$$

$$f_8 = \text{Simplify}\left[\frac{\text{temp}}{\sqrt{\text{length}[\text{temp}]}}\right]$$

$$\text{temp} = \text{Simplify}\left[e_9 - \sum_{j=0}^{3} \text{Leng}[f_{2j+1}, e_9]\, f_{2j+1}\right];$$

$$f_9 = \text{Simplify}\left[\frac{\text{temp}}{\sqrt{\text{length}[\text{temp}]}}\right]$$

$$\frac{1}{\sqrt{2}}$$

$$\sqrt{\frac{3}{2}}\; x$$

$$\frac{1}{2}\sqrt{\frac{5}{2}}\; (-1 + 3 x^2)$$

$$\frac{1}{2}\sqrt{\frac{7}{2}}\; x\, (-3 + 5 x^2)$$

$$\frac{3\,(3 - 30 x^2 + 35 x^4)}{8\sqrt{2}}$$

$$\frac{1}{8}\sqrt{\frac{11}{2}}\; x\, (15 - 70 x^2 + 63 x^4)$$

$$\frac{1}{16}\sqrt{\frac{13}{2}}\left(-5+105\,x^2-315\,x^4+231\,x^6\right)$$

$$\frac{1}{16}\sqrt{\frac{15}{2}}\,x\left(-35+315\,x^2-693\,x^4+429\,x^6\right)$$

$$\frac{1}{128}\sqrt{\frac{17}{2}}\left(35-1260\,x^2+6930\,x^4-12012\,x^6+6435\,x^8\right)$$

$$\frac{1}{128}\sqrt{\frac{19}{2}}\,x\left(315-4620\,x^2+18018\,x^4-25740\,x^6+12155\,x^8\right)$$

The matrix X represents the linear operation which corresponds to multiplying by x.

```
X = Table[Leng[f_i, x f_j], {i, 0, 9}, {j, 0, 9}];
X // MatrixForm
```

$$\begin{pmatrix}
0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{15}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{2}{\sqrt{15}} & 0 & \frac{3}{\sqrt{35}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{3}{\sqrt{35}} & 0 & \frac{4}{3\sqrt{7}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{4}{3\sqrt{7}} & 0 & \frac{5}{3\sqrt{11}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{5}{3\sqrt{11}} & 0 & \frac{6}{\sqrt{143}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{6}{\sqrt{143}} & 0 & \frac{7}{\sqrt{195}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{\sqrt{195}} & 0 & \frac{8}{\sqrt{255}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{8}{\sqrt{255}} & 0 & \frac{9}{\sqrt{323}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{\sqrt{323}} & 0
\end{pmatrix}$$

The matrix Dx represents the linear operation which corresponds to differentiating by x. The catch is that it is not anti−hermitian. Dxx that we construct later is made anti−hermitian by hand. It has the right commutation relation with X as well (modulo edge effects corresponding to our truncation to finite number of basis vectors).

```
Dx = Table[Leng[fj, ∂x fi], {j, 0, 9}, {i, 0, 9}];
Dx // MatrixForm
```

$$\begin{pmatrix}
0 & \sqrt{3} & 0 & \sqrt{7} & 0 & \sqrt{11} & 0 & \sqrt{15} & 0 & \sqrt{19} \\
0 & 0 & \sqrt{15} & 0 & 3\sqrt{3} & 0 & \sqrt{39} & 0 & \sqrt{51} & 0 \\
0 & 0 & 0 & \sqrt{35} & 0 & \sqrt{55} & 0 & 5\sqrt{3} & 0 & \sqrt{95} \\
0 & 0 & 0 & 0 & 3\sqrt{7} & 0 & \sqrt{91} & 0 & \sqrt{119} & 0 \\
0 & 0 & 0 & 0 & 0 & 3\sqrt{11} & 0 & 3\sqrt{15} & 0 & 3\sqrt{19} \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{143} & 0 & \sqrt{187} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{195} & 0 & \sqrt{247} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{255} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{323} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

```
Dxx = (Dx - Transpose[Dx]) / 2;
Dxx // MatrixForm
```

$$\begin{pmatrix}
0 & \frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{7}}{2} & 0 & \frac{\sqrt{11}}{2} & 0 & \frac{\sqrt{15}}{2} & 0 & \frac{\sqrt{19}}{2} \\
-\frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{15}}{2} & 0 & \frac{3\sqrt{3}}{2} & 0 & \frac{\sqrt{39}}{2} & 0 & \frac{\sqrt{51}}{2} & 0 \\
0 & -\frac{\sqrt{15}}{2} & 0 & \frac{\sqrt{35}}{2} & 0 & \frac{\sqrt{55}}{2} & 0 & \frac{5\sqrt{3}}{2} & 0 & \frac{\sqrt{95}}{2} \\
-\frac{\sqrt{7}}{2} & 0 & -\frac{\sqrt{35}}{2} & 0 & \frac{3\sqrt{7}}{2} & 0 & \frac{\sqrt{91}}{2} & 0 & \frac{\sqrt{119}}{2} & 0 \\
0 & -\frac{3\sqrt{3}}{2} & 0 & -\frac{3\sqrt{7}}{2} & 0 & \frac{3\sqrt{11}}{2} & 0 & \frac{3\sqrt{15}}{2} & 0 & \frac{3\sqrt{19}}{2} \\
-\frac{\sqrt{11}}{2} & 0 & -\frac{\sqrt{55}}{2} & 0 & -\frac{3\sqrt{11}}{2} & 0 & \frac{\sqrt{143}}{2} & 0 & \frac{\sqrt{187}}{2} & 0 \\
0 & -\frac{\sqrt{39}}{2} & 0 & -\frac{\sqrt{91}}{2} & 0 & -\frac{\sqrt{143}}{2} & 0 & \frac{\sqrt{195}}{2} & 0 & \frac{\sqrt{247}}{2} \\
-\frac{\sqrt{15}}{2} & 0 & -\frac{5\sqrt{3}}{2} & 0 & -\frac{3\sqrt{15}}{2} & 0 & -\frac{\sqrt{195}}{2} & 0 & \frac{\sqrt{255}}{2} & 0 \\
0 & -\frac{\sqrt{51}}{2} & 0 & -\frac{\sqrt{119}}{2} & 0 & -\frac{\sqrt{187}}{2} & 0 & -\frac{\sqrt{255}}{2} & 0 & \frac{\sqrt{323}}{2} \\
-\frac{\sqrt{19}}{2} & 0 & -\frac{\sqrt{95}}{2} & 0 & -\frac{3\sqrt{19}}{2} & 0 & -\frac{\sqrt{247}}{2} & 0 & -\frac{\sqrt{323}}{2} & 0
\end{pmatrix}$$

```
Simplify[Dx. X – X . Dx] // MatrixForm
Simplify[Dxx. X – X . Dxx] // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -10\sqrt{\frac{3}{19}} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -10\sqrt{\frac{7}{19}} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -10\sqrt{\frac{11}{19}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -10\sqrt{\frac{15}{19}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -9 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5\sqrt{\frac{3}{19}} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -5\sqrt{\frac{7}{19}} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -5\sqrt{\frac{11}{19}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -5\sqrt{\frac{15}{19}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -5\sqrt{\frac{3}{19}} & 0 & -5\sqrt{\frac{7}{19}} & 0 & -5\sqrt{\frac{11}{19}} & 0 & -5\sqrt{\frac{15}{19}} & 0 & -9 \end{pmatrix}$$