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Wave Motion

Contents Laplace's and Poisson's equations, harmonic functions, the diffusion equation, the wave equation.

1 The Laplacian operator

We have seen that the del operator $\vec{\nabla}$ plays a fundamental role in the analysis of a very large number of physical phenomena in various fields. It leads naturally

- from a scalar field ϕ to a vector field, namely, the gradient $\vec{\nabla}\phi$;
- from a vector field $\vec{A}$ to a scalar field, namely, the divergence $\vec{\nabla} \cdot \vec{A}$; and
- from a vector field $\vec{A}$ to another vector field, namely, the curl $\vec{\nabla} \times \vec{A}$.

You may have encountered several applications of these concepts as they pertain to more than one area of physics, such as particle mechanics, fluid dynamics, and electromagnetism in elementary courses such as Physics I and II.

The gradient, divergence and curl are all quantities that require a **single** application of the operator $\vec{\nabla}$ to scalar or vector functions, and hence they involve **first** derivatives (with respect to the coordinates) of the fields concerned. There is another operator that can be constructed using the $\vec{\nabla}$ operator that is of fundamental importance in the physical sciences. This is the **second order** differential operator $\vec{\nabla} \cdot \vec{\nabla}$ that is written as ∇^2, and called "del squared". It is also referred to as the Laplacian operator, or as simply the **Laplacian**. Since $\vec{\nabla}$ is a **vector** differential operator, and ∇^2 is the dot product of $\vec{\nabla}$ with itself, the Laplacian operator is a **scalar** differential operator. It is given in Cartesian coordinates by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \equiv \partial_i \, \partial_i \,, \tag{1}$$

where the RHS of the final equation expresses ∇^2 in the index notation introduced earlier. (Recall that the repeated index i is to be summed over the values $1, 2$ and 3 corresponding, respectively, to the three Cartesian coordinates.)

Since ∇^2 is itself a scalar operator, when it acts on a scalar field $\phi\,(\vec{r})$, the result is again a scalar field:

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \,. \tag{2}$$

Similarly, when it acts on a vector field $\vec{A}$, the result is another vector field. In Cartesian coordinates, we have

$$\nabla^2 \vec{A} = (\nabla^2 A_x)\,\hat{e}_x + (\nabla^2 A_y)\,\hat{e}_y + (\nabla^2 A_z)\,\hat{e}_z\,, \tag{3}$$

where, of course,

$$\nabla^2 A_x = \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\,, \tag{4}$$

and similarly for $\nabla^2 A_y$ and $\nabla^2 A_z$.

We may ask whether a second-order **vector** differential operator can be constructed from the $\vec{\nabla}$ operator. The obvious candidate is curl grad (acting on any scalar field ϕ), but this is identically zero. We note, too, that the other possible second order operator, div curl (acting on any vector field $\vec{A}$), also vanishes identically.

2 Occurrence of the Laplacian in physical situations

Let us now list the most common physical contexts in which ∇^2 appears.

• **Poisson's equation** Recall that an electrostatic field $\vec{E}(\vec{r})$ satisfies the equations

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0} \quad \text{and} \quad \vec{\nabla} \times \vec{E} = 0\,. \tag{5}$$

The fact that it is an irrotational field implies that any electrostatic field can always be written as the gradient of a scalar field. By convention, this is minus the electrostatic potential $\phi\,(\vec{r})$. Thus $\vec{E} = -\vec{\nabla}\Phi$. When this is substituted in Eq. (5), we get

$$\nabla^2\ \Phi(\vec{r}) = -\frac{\rho(\vec{r})}{\epsilon_0}\,. \tag{6}$$

This is called **Poisson's equation**. It shows how the charge density acts as a **source** for the electrostatic potential, and hence for the electrostatic field. To obtain the potential from this second order partial differential equation, we must solve it with appropriate boundary conditions. Once such boundary conditions are specified, the solution to the equation is **unique**. For instance, if we are solving for Φ in all space, i.e., $\mathbb{R}^3$, the one may impose the boundary condition that Φ vanishes at spatial infinity: $\lim_{r\to\infty}\Phi = 0$.

• **Laplace's equation** In a region in which there are no charges or charge density present, Poisson's equation reduces to

$$\nabla^2\ \Phi = 0\,. \tag{7}$$

This is called **Laplace's equation**. It appears in many other contexts as well. For instance, in a region in which there is no mass density present, the **gravitational potential** also satisfies Laplace's equation.

The same equation appears in fluid dynamics as well. The velocity field in the irrotational flow of an incompressible, inviscid fluid satisfies the equations

$$\vec{\nabla} \cdot \vec{v} = 0 \quad \text{and} \quad \vec{\nabla} \times \vec{v} = 0 \,. \tag{8}$$

Thus $\vec{v}$ can be written as the gradient of a scalar field, or $\vec{v} = -\vec{\nabla}\phi$. The scalar field ϕ in this case is called the **velocity potential**, by analogy with the instances mentioned earlier. (Once again, the minus sign is just a matter of convention.) The velocity potential then satisfies Laplace's equation, $\nabla^2\phi = 0$.

• **The Helmholtz equation** When many physical systems are set in vibration or oscillation, their natural modes of vibration can be described by suitable sets of standing waves. The standing waves in a string clamped at both ends, those in a membrane or drumhead set in vibration, sound waves a resonating cavity, waves on the surface of a liquid, stress waves in solids, etc., are just a few examples out of very many. Such waves occur as solutions of the **Helmholtz equation**

$$(\nabla^2 + k^2)\phi = 0 \,, \tag{9}$$

where k represents the wavenumber ($= 2\pi/\lambda$ where λ is the wavelength). The possible values of k depend on the boundary conditions imposed in each case.

The instances listed above deal with either static phenomena or steady-state phenomena, and therefore do not involve any time derivatives. But there are also very fundamental time-dependent phenomena in which ∇^2 appears naturally. The two most important cases in this regard are listed below.

• **The diffusion or heat conduction equation** Diffusion is the process by which an uneven concentration of a substance gets gradually smoothed out spontaneously – e.g., a concentration of a chemical species (like a drop of ink) in a beaker of water spreads out "by itself", even in the absence of stirring. The detailed mechanism of diffusion involves the random collisions that take place at the molecular level, but a macroscopic description of the process can be given on simple physical grounds. We shall do so shortly. For the present we merely introduce the equation that describes diffusion. Let $\rho\,(\vec{r}, t)$ denote the concentration of the substance of interest, at the point $\vec{r}$ and at time t. Then it can be shown that $\rho\,(\vec{r}, t)$ satisfies the **diffusion equation**

$$\frac{\partial \rho\,(\vec{r}, t)}{\partial t} = D\,\nabla^2\,\rho\,(\vec{r}, t) \,, \tag{10}$$

where D is called the **diffusion constant** or **diffusion coefficient**.

Exactly the same equation is obtained in the problem of **heat conduction**, with the concentration field $\rho\,(\vec{r}, t)$ replaced by the temperature field $T(\vec{r}, t)$, and the diffusion constant D replaced by the **thermal conductivity** κ . The equation giving the temperature at any point of a thermally conducting body whose initial temperature distribution is specified, and whose surface is maintained at some given temperature distribution, is

$$\frac{\partial T\,(\vec{r},t)}{\partial t} = \kappa\,\nabla^2\,T\,(\vec{r},t)\,. \tag{11}$$

For this reason, the diffusion equation is also called the **heat equation**.

• **The wave equation** We have stated that the Helmholtz equation describes standing waves in various physical systems. Such waves are special cases of more general wave motion, that also includes travelling waves of different kinds – pulses, sinusoidal waves, etc. The simplest and most basic equation that describes wave motion is the **wave equation**,

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\,\phi = 0\,. \tag{12}$$

The constant v is called the wave velocity (it is actually the speed with which the wavefronts move). Equation (12) pertains to scalar waves, in the sense that it describes wave-like variation of a scalar field ϕ. Similarly, vector fields can also have wave-like variation – the most important example being electromagnetic waves. In this case the electric and magnetic fields oscillate in space and time. In free space, for instance, they satisfy the wave equations

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\,\vec{E} = 0 \quad \text{and} \quad \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\,\vec{B} = 0\,, \tag{13}$$

where c is the speed of light in a vacuum. It is equal to $1/\sqrt{\mu_0\epsilon_0}$, where μ_0 and ϵ_0 are the permeability and permittivity, respectively, of free space.

We now consider the equations introduced above in a little more detail, in order to bring out some of the physically significant features of their solutions.

3 Laplace's equation and harmonic functions

Any solution of Laplace's equation $\nabla^2\phi = 0$ is called a **harmonic function**. To understand what is so special about such functions, it is helpful to consider, first, the trivial case of a harmonic function that only depends on one coordinate, say x. Laplace's equation then reduces to $d^2\phi/dx^2 = 0$, whose general solution is the linear function $\phi\,(x) = ax + b$. The constants a and b are fixed by the boundary conditions. The graph of $\phi\,(x)$ is just a straight line.

Although this is an almost trivial case, two of its features are significant. First, the value of $\phi\,(x)$ at any given point x_0 is the **arithmetic mean** of its values at any two points symmetrically located on either side of x. That is,

$\phi(x_0) = \frac{1}{2}[\phi(x_0 - c) + \phi(x_0 + c)]$ for any real c. It is obvious that this is a direct consequence of the absence of any **curvature** in the graph of the linear function $\phi(x)$. The second feature is a consequence of the first. In any interval $[x_0 - c, x_0 + c]$, the largest value and the least value of $\phi(x)$ can only occur at the **boundaries** or end points $(x_0 - c)$ and $(x_0 + c)$, and not at any point in the **interior** of the interval.

These features carry over to all harmonic functions, i.e., to all solutions of Laplace's equation in an arbitrary number of dimensions. In two and more dimensions, harmonic functions are **not** restricted to linear functions of the coordinates – far from it. In fact, harmonic functions in two and more dimensions can be quite intricate, and exhibit a rich variety of properties. In two dimensions, for instance, a harmonic function satisfies the equation

$$\nabla^2 \phi(x, y) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0\,. \tag{14}$$

Linear functions like $ax + by + c$ continue to be solutions of this equation. But nonlinear functions can also be solutions. For example, $\phi(x, y) = xy$ is such a solution. So is the function $\phi(x, y) = (x^2 - y^2)$. And we know that the electrostatic potential due to a point charge at the origin, namely $1/r$ (aside from a multiplicative constant), satisfies Laplace's equation in three dimensions for all $r \neq 0$. It is thus a harmonic function in the region $r > 0$.

More generally, we have seen that the real and imaginary parts of any holomorphic function (or anti-holomorphic function) are harmonic functions in two-dimensions. We have also seen that **a harmonic function cannot have maxima or minima in the interior of a region** (i.e., in an open region); such points can therefore only occur at the **boundary** of the region in which the function concerned is harmonic. For, suppose the harmonic function $\phi(x, y)$ has a simple maximum [or minimum] at a point in the interior of a region in which it is harmonic. Then both $\partial^2\phi/\partial x^2$ and $\partial^2\phi/\partial y^2$ must be negative [respectively, positive] at that point. But this is in contradiction with Eq. (14).

A direct physical consequence of this is the fact that the electrostatic potential $\phi(\vec{r})$ cannot have a mximum or minimum in any charge-free region. Therefore a test charge placed at any such point **cannot be in stable equilibrium**, because this would require that Φ have a **minimum** at that point. This result is the basis of **Earnshaw's Theorem**, which states: **A set of charges held together by electrostatic forces alone cannot be in a state of stable equilibrium**.

It is important to note that this does **not** forbid points of **unstable** equilibrium. Taking the two-dimensional example mentiond above, $\phi(x, y) = x^2 - y^2$, we see that the origin (0, 0 is neither a maximum nor minimum of ϕ, but rather a **saddle point**. On a disc of radius R centred at the origin, the maximum value of ϕ occurs at the points $(\pm R, 0)$, and the minimum value occurs at the points $(0, \pm R)$, all of which lie on the boundary of the disc.

We have also seen that **the value of a harmonic function at any point** P **is the mean of its values on the boundary of a region centred about** P. In the example above, the boundary is the circumference of the circle $x^2 + y^2 = R^2$. Therefore the mean value concerned is just

$$\frac{1}{2\pi R}\oint dl\,(x^2 - y^2) = \frac{1}{2\pi R}\int_0^{2\pi} R\,d\theta\,(R^2\,\cos^2\,\theta - R^2\,\sin^2\,\theta) = 0\,, \tag{15}$$

which is the value of ϕ at the origin. Once again, this result has an implication in electrostatics: the electrostatic potential at any point $\vec{r}_0$ in a charge-free region is the mean value of the potential on a sphere of arbitrary radius R centred at $\vec{r}_0$, provided there are no charges on or inside the sphere:

$$\Phi(\vec{r}_0) = \frac{1}{4\pi R^2}\int_S dS\,\phi\,(\vec{r})\,, \tag{16}$$

where $\vec{r}$ runs over the surface of the sphere.

Another crucial property of Laplace's equation (and, indeed, of all the equations we have listed above) is that it is a **linear** equation. Therefore the **principle of superposition** holds good: any linear combination of independent solutions is also a solution to the equation. This enables us to superpose different harmonic functions to obtain the solution that satisfies the given boundary conditions. This is the basis of **the method of images**.

4 The diffusion or heat equation

We have already written down the diffusion equation, Eq. (10). Let us now see how it is obtained based on simple physical arguments. Recall that $\rho(\vec{r}, t)$ denotes the instantaneous concentration of the diffusing substance at the point $\vec{r}$.

In the absence of any sources or sinks of the substance, conservation of matter implies the equation of continuity which reads

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J}\,(\vec{r}, t) = 0\,. \tag{17}$$

The corresponding "diffusion current" $\vec{J}$ is yet to be specified. The equation of continuity is sometimes called **Fick's First Law** in the context of diffusion. The next step is the crucial physical input: one assumes that **the diffusion current is proportional to the local difference in concentrations, i.e., to the negative of the gradient of the concentration itself**. This assumption is valid quite generally, unless the concentration gradients are very large. The minus sign arises because diffusion entails a flow of the substance from a **higher** concentration to a **lower** concentration. The constant of proportionality is the diffusion constant D. Thus

$$\vec{J}\,(\vec{r}, t) = -D\,\vec{\nabla}\rho\,(\vec{r}, t)\,. \tag{18}$$

This is **Fick's Second Law**, sometimes simply called Fick's Law. The physical dimensions of D are $L^2 T^{-1}$.

Taking the divergence of both sides of Eq. (18) and substituting the result in Eq. (17) immediately yields Eq. (10), the diffusion equation. This is again a linear equation. It is of second order in the coordinates, but of first order in time. To find a unique solution to the diffusion equation in a given region, we must specify the initial concentration $\rho(\vec{r}, 0)$, as well as appropriate boundary conditions.

Exercise Verify that the expression $\rho(\vec{r}, t) = (4\pi D t)^{-3/2} \exp(-r^2/4Dt)$ satisfies Eq. (10). This is the basic solution to the diffusion equation in three dimensions, from which specific solutions can be constructed to suit various boundary conditions.

The heat conduction equation, Eq. (11), is derived in a similar manner. The equation of continuity is now used to express the conservation of energy rather than mass. The counterpart of Fick's Second Law is the assumption that **the heat flux is proportional to the negative of the temperature gradient**.

The fact that Eqs. (10) and (11) involve the first derivative with respect to time is of physical significance. This arises essentially because diffusion and heat conduction are **irreversible** and **dissipative** processes. A drop of ink, once it spreads out throughout a beaker of water, does not spontaneously reconstitute itself as a drop of ink no matter how long the system is kept under observation. Nor does heat flow from the cold end to the hot end of a rod whose ends are at different temperatures.

5 The Helmholtz equation and normal modes

The natural modes of vibration or oscillation of a system are called its **normal modes**. As already mentioned, the Helmholtz equation (Eq. (9)) describes this kind of motion.

The general method of solution of Eq. (9) is discussed at the end of this handout. However, one particular feature of the problem is very important: namely, the fact that the **boundary conditions play a crucial role in determining the allowed normal modes of a system**. Fortunately, the relatively simple one-dimensional case suffices to illustrate this assertion.

Consider a stretched string along the x-axis, with its ends clamped at $x = 0$ and $x = L$. Its transverse displacement **at any fixed instant of time** satisfies the Helmholtz equation. In one dimension, this is just

$$\frac{d^2\phi(x)}{dx^2} + k^2 \phi(x) = 0\,. \qquad (19)$$

As the ends of the string are clamped, the appropriate boundary conditions are $\phi(0) = 0$ and $\phi(L) = 0$. The general solution of Eq. (19) is $\phi(x) = A \sin kx + B \cos kx$ where A and B are constants. Imposing the boundary condition $\phi(0) = 0$ shows that B must be zero. Imposing the boundary condition $\phi(L) = 0$ implies that either A is zero, in which case there is no nonzero solution at all; or else k must satisfy the condition $\sin kL = 0$, so that $k = n\pi/L$ where n is an integer. In turn, this means that **the wavelength of the waves is restricted to the set of values $\lambda = 2L/n$ where $n = 1, 2, \ldots$.** The physically acceptable solutions to Eq. (19) in the present case are therefore given by

$$\phi(x) = A \sin\left(\frac{n\pi x}{L}\right), \quad \text{where} \quad n = 1, 2, \ldots. \tag{20}$$

Note that negative values of n do not give **independent** solutions, because $\sin(-n\pi x/L)$ is just (-1) times $\sin(n\pi x/L)$. Hence the independent soultions are just the ones corresponding to positive integral values of n.

We have seen that the **wavelength** gets selected by the boundary conditions. What about the frequency? The speed of waves is a property of the medium concerned. In the present example, it is fixed by the tension and mass per unit length of the string. Once the speed and wavelength are given, the frequency is of course determined by their ratio.

The example above shows clearly how **the boundary conditions select the admissible natural modes of vibration**. Exactly the same arguments go through in higher dimensions as well. For example, consider the normal modes of vibration of a rectangular membrane or drumhead of sides L_1 and L_2, with all its edges clamped. Let the corners of the membrane have coordinates $(0, 0)$, $(0, L_1)$, (L_1, L_2) and $(0, L_2)$. The transverse displacement of the membrane at any fixed instant of time, $\phi(x, y)$, satisfies the Helmholtz equation in two dimensions. The physically acceptable solutions in this case are given by

$$\phi(x, y) = A \sin\left(\frac{n_1 \pi x}{L_1}\right) \sin\left(\frac{n_2 \pi y}{L_2}\right), \tag{21}$$

where A is a constant and $n_1,\, n_2 = 1, 2, \ldots$. The normal modes in this instance are thus specified by a **pair** of positive integers, n_1 and n_2. The extension of this solution to the case of standing waves in a rectangular parallelepiped is obvious.

The relative simplicity of the solution in Eq. (21) is actually due to the Cartesian symmetry of the problem. The shape of the drumhead is rectangular, and the ∇^2 operator is just the sum of the individual second derivatives with respect to x and y. Even as symmetrical a case as a **circular** drumhead leads to considerably more complicated solutions. Consider the case of a circular membrane of radius R, clamped all along its circumference and centred at the origin. Helmholtz's equation in plane polar coordinates (ϱ, φ) is

$$\frac{1}{\varrho}\frac{\partial}{\partial \varrho}\left(\varrho \frac{\partial \phi}{\partial \varrho}\right) + \frac{1}{\varrho^2}\frac{\partial^2 \phi}{\partial \varphi^2} + k^2 \phi = 0. \tag{22}$$

Some simplification occurs in the case of circularly symmetric vibrations, for which ϕ is a function of the radial distance ϱ alone. Equation (22) then reduces to the ordinary differential equation

$$\frac{d^2\phi}{d\varrho^2} + \frac{1}{\varrho}\frac{d\phi}{d\varrho} + k^2\,\phi = 0\,, \tag{23}$$

to be solved with the boundary condition $\phi\,|_{\varrho=R} = 0$. The solution turns out to be a **Bessel function** of the zeroth order and first kind: $\phi\,(\varrho) = A\,J_0(k\varrho)$ where A is a constant. Once again, k is restricted to an infinite discrete set of values given by the solutions of the equation $J_0(kR) = 0$. This can only be solved numerically. We find $kR = 2.404,\ 5.520,\ 8.653, \ldots$.

We mention in passing that the general problem of finding the normal modes of a system from the Helmholtz equation together with appropriate boundary conditions is an example of what is known as the **eigenvalue problem**.

6 Solution of the wave equation in one dimension

Finally, we turn to the wave equation. We restrict ourselves here to scalar waves, given by Eq. (12).

First, consider the connection between the wave equation and the Helmholtz equation. We have stated that the former is applicable for both travelling waves as well as standing waves, while the latter involves no time derivatives. Let v denote, as usual, the speed of the waves concerned (this is a property of the medium, as already mentioned). If we consider waves with an angular frequency ω, the time dependence of ϕ involves the sine or cosine of ωt. The term $(1/v^2)\,(\partial^2\phi/\partial t^2)$ in the wave equation then reduces to $-(\omega^2/v^2)\,\phi$, so that the wave equation reduces to Helmholtz's equation for the coordinate-dependent part of $\phi\,(\vec{r}, t)$, **with the identification** $\omega/v = k$. This is nothing but the statement that the corresponding wavelength is given by $\lambda = 2\pi/k = 2\pi v/\omega = v/\nu$, where ν is the frequency.

The general solution of Eq. (12) is discussed at the end of this handout. Our purpose here is to bring out some of the important physical features of the solution. This can be done to some extent even in the simpler case of the one-dimensional wave equation. We shall therefore consider this case in some detail, and subsequently make some relevant remarks about the higher-dimensional cases.

The one-dimensional wave equation is

$$\frac{1}{v^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = 0\,. \tag{24}$$

To solve this equation, it is very convenient to make a change of independent variables from (x, t) to the pair (ξ, η), where $\xi = (x - vt)$ and $\eta = (x + vt)$.

Then, after simplification, Eq. (24) reduces to

$$\frac{\partial^2 \phi}{\partial \xi \, \partial \eta} = 0 \,. \tag{25}$$

Exercise Verify this. You will need to use standard chain rule formulas such as

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \phi}{\partial \eta}\frac{\partial \eta}{\partial x} \quad \text{and} \quad \frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial \phi}{\partial \eta}\frac{\partial \eta}{\partial t} \,,$$

and similar formulas for the second derivatives.

It is easy to solve Eq. (25). We have

$$\frac{\partial}{\partial \xi}\left(\frac{\partial \phi}{\partial \eta}\right) = 0 \,, \tag{26}$$

which is readily integrated to give $\partial\phi/\partial\eta$ = some function of η alone, say $G(\eta)$ (rather than a constant, as in the case of an ordinary derivative). On the other hand, Eq. (25) can also be written as

$$\frac{\partial}{\partial \eta}\left(\frac{\partial \phi}{\partial \xi}\right) = 0 \,, \tag{27}$$

which can be integrated to give $\partial\phi/\partial\xi$ = some function of ξ alone, say $F(\eta)$ (rather than a constant). Therefore the general solution **must** be of the form

$$\phi\,(\xi, \eta) = \int F(\xi)\, d\xi + \int G(\eta)\, d\eta \equiv f(\xi) + g(\eta) \,, \tag{28}$$

so that both Eq. (26) and Eq. (27) are satisfied. Going back to the original variables x and t, this means that the most general solution of the wave equation in one dimension is given by

$$u(x,t) = f(x - vt) + g(x + vt) \,, \tag{29}$$

where f and g are arbitrary twice-differentiable functions. (Any additive constant can be included in f or g.)

7 Physical interpretation of the solution

At $t = 0$, suppose we sketch the graph of $f(x)$ versus x. Let $f(x_0) = f_0$ at $t = 0$. Then, at the instant time $t = t_1 > 0$, this value occurs not at $x = x_0$, but at the point $x_1 = x_0 + vt_1$. Thus the graph of $f(x - vt)$ at any later time $t > 0$ is obtained by simply sliding the original graph to the right by the amount vt. Hence $f(x - vt)$ represents a **wave** travelling to the **right** with speed v. Note that $f(x)$ **need not** be a sine or cosine function, or any other oscillatory function. **A wave is just a disturbance, at any instant, of any point of a medium travelling through the medium as time elapses.** If the disturbance is given by a sine or cosine function of $(x-vt)$, then we call it a **sinusoidal** or **harmonic wave**. In general, $f(x-vt)$ represents a progressive or travelling wave for any profile $f(x)$.

In exactly the same fashion, $g(x+vt)$ represents a wave profile $g(x)$ travelling to the **left** on the x-axis, with a speed v. Once again, $g(x)$ need not be an oscillatory function of its argument. The general solution in Eq. (29) is a superposition of a forward moving profile $f(x)$ and a backward moving profile $g(x)$. The speed v of these waves is a constant that depends only on the properties of the medium supporting these waves. For instance, in the case of a travelling wave on a taut string, it depends on the tension and mass density of the string. It does not depend on the specific forms of $f(x)$ and $g(x)$. The function $f(x-vt)$ itself can be the superposition of several different functions of $(x-vt)$; likewise for $g(x+vt)$. The superposition principle holds good because the wave equation is a linear equation in ϕ. The precise forms of $f(x)$ and $g(x)$ are determined by the initial conditions and boundary conditions imposed on $\phi\,(x,t)$.

8 Harmonic waves

In one dimension, harmonic waves are functions of the form $\phi\,(x,t) = A\,\cos\,(kx \pm \omega t)$ or $B\,\sin\,(kx \pm \omega t)$. Such functions are solutions of the wave equation, Eq. (24), provided the relation

$$\omega = v\,k \tag{30}$$

is satisfied. Note that it is notationally simpler to work always with the angular frequency ω and the wavenumber k, rather than the frequency $\nu\,(=\omega/2\pi)$ and the wavelength $\lambda\,(=2\pi/k)$. In physical media, the speed of a wave is generally dependent on its wavelength. A wave-packet or pulse made up of harmonic waves of several wavelengths spreads out or disperses as it propagates in the medium. The general relation $\omega = \omega\,(k)$ specifying the frequency as a function of the wavelength is called a **dispersion relation**. Equation (30) is the simplest possible relation of this kind.

The generalization of the idea of harmonic waves to two and three dimensions is straightforward. The argument $(kx - \omega t)$ of the sine or cosine function is replaced by $(\vec{k}\cdot\vec{r} - \omega t)$, where $\vec{k}$ is called the wave vector. Its direction represents the direction of propagtion of the wave front, while its magnitude gives the wavelength according to $2\pi/|\vec{k}| = \lambda$.

Exercise Verify that $\phi\,(\vec{r},t) = A\,\cos\,(\vec{k}\cdot\vec{r} - \omega t)$ is a solution of the wave equation, Eq. (12), provided the condition $\omega = v\,|\vec{k}|$ is satisfied.

The importance of harmonic waves arises due to the following facts:

- They represent disturbances with a definite wavelength $\lambda = 2\pi/k$ and frequency $\nu = \omega/2\pi$.
- Subject to certain very general conditions, **any** periodically varying function of $(\vec{r}, t)$ can be written as a linear combination or superposition of harmonic waves. This is called a **Fourier series** expansion. Even functions that do **not** oscillate or vary periodically can be written in this form. This is called a **Fourier transform** or Fourier integral representation.

9 The general method

Let $\mathcal{L}$ be a linear self-adjoint operator acting on some Hilbert space $\mathcal{H}$. In order to have a concrete example, let $\mathcal{L} = -\nabla^2$ and $\mathcal{H}$ be the space of box normalisable functions, i.e., functions ϕ such that

$$\int_{\text{cubical box of size } L} d^3x \qquad |\phi|^2 < \infty \ .$$

Let the values of $x_i \in [-L/2, L/2]$. One solves the following eigenvalue problem

$$\mathcal{L}\ \phi_\lambda = \lambda\ \phi_\lambda \ .$$

and finds all solutions ϕ_λ (with eigenvalue λ) that satisfy appropriate boundary conditions[1] Since $\mathcal{L}$ is a self-adjoint operator, the eigenvalues will be real.

In our example, two possible boundary conditions are: (i) require that $\phi = 0$ on the boundary of the box; (ii) impose periodic boundary conditions; $\phi(x_1, x_2, x_3) = \phi(x_1 + L, x_2, x_3) = \phi(x_1, x_2 + L, x_3) = \phi(x_1, x_2, x_3 + L)$. Since boundary condition (i) is similar to what we discussed earlier, let us consider the case of periodic boundary conditions. It is not hard to see that the complete of eigenfunctions (with eigenvalue $k_{\vec{n}}^2 = 4\pi^2(n_1^2 + n_2^2 + n_3^2)/L^2$) are

$$\phi_{\vec{n}}(\vec{x}) = \frac{1}{\sqrt{L^3}} \exp\left[\frac{ik_i x_i}{L}\right] = \frac{1}{\sqrt{L^3}} \exp\left[\frac{i2\pi n_i x_i}{L}\right] , \tag{31}$$

where n_1, n_2, $n_3 \in \mathbb{Z}$.
Exercise: Extend the above method to formally solve Poisson's equation: $-\nabla^2\Psi(\vec{x}) = 4\pi\rho(\vec{x})$, where $\rho(\vec{x})$ is a source and show that

$$\Psi(\vec{x}) = 4\pi \sum_{\vec{n}\neq\vec{0}} \frac{\rho_{\vec{n}}}{k_{\vec{n}}^2}\ \phi_{\vec{n}}(\vec{x}) + \text{harmonic pieces} \ ,$$

where $\rho_{\vec{n}}$ is defined through the expansion $\rho(\vec{x}) = \sum_{\vec{n}} \rho_{\vec{n}}\ \phi_{\vec{n}}(\vec{x})$.

[1] Boundary conditions are important – in fact, boundary conditions are necessary in deciding whether a given operator is self-adjoint.