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PH350 Classical Physics **Handout 2** **27.8.2009**

Lie Groups and Lie Algebras

We have seen that the exponential map provides an interesting map relating the special orthogonal group $SO(n)$ and real anti-symmetric $n \times n$ matrices. A similar relationship appears for the unitary group $U(n)$ and anti-hermitian $n \times n$ matrices[1]. One has

$$O \longleftrightarrow \exp(A) \quad , \quad U \longleftrightarrow \exp(iH) \ ,$$

for $n \times n$ special orthogonal and unitary matrices with A (H) is a real anti-symmetric (hermitian) matrix.

Mathematically, *a Lie group G is a continuous group where the elements of G form a smooth space ("manifold") and the group action is also smooth.* The special orthogonal groups, special unitary groups and symplectic groups are important examples of Lie groups that appear in many physical applications.

Exercise: Show that the 2×2 matrix U

$$U = \alpha_0\ I_2 + i \sum_{j=1}^{3}\ \alpha_j\ \sigma^j \quad ,$$

where σ^i are the Pauli sigma matrices[2] and α_0, α_i are real parameters is a $SU(2)$ matrix when

$$\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1 \ .$$

This is the equation of a *three-dimensional sphere* S^3 of unit radius. In other words, every point on the sphere corresponds to an element of the Lie group $SU(2)$. One says that '*the three-sphere is the group manifold for Lie group $SU(2)$*'.

The exponential map relates Lie groups to Lie algebras. Before embarking on the definition of a Lie algebra, let us study $SO(n)$ in some more detail.

- An important observation is that the set of $n \times n$ anti-symmetric matrices form a real linear vector space of dimension $n(n-1)/2$.
- The product of two anti-symmetric matrices A and B is **not** antisymmetric. However, the commutator of two anti-symmetric matrices, $[A, B]$, is

[1] Note that any anti-hermitian matrix can be represented as i times an hermitian matrix.
[2] The Pauli sigma matrices are: $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

anti-symmetric (check this). Now consider the *Baker-Campbell-Hausdorff* (BCH) formula for the multiplication of two matrix exponentials

$$\begin{aligned} e^A \, e^B &= e^C \quad , \text{ where} \\ C &= A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] - \frac{1}{12}[B,[A,B]] + \cdots \end{aligned}$$

where the ellipsis corresponds to higher commutators involving the matrices A and B. When A and B are antisymmetric matrices, it is not hard to see that C is anti-symmetric as well. This can also be obtained from the closure property of $SO(n)$.

- The Jacobi identity

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0 \ ,$$

is satisfied as well.

These properties define a Lie algebra. A more precise definition is as follows: *A Lie algebra is a linear vector space equipped with a skew-symmetric bilinear product (called the Lie bracket) which satisfies the Jacobi identity.* In our realisation of the Lie algebra, the Lie bracket is given by the commutator. It is conventional to represent the Lie algebra of a Lie group G by the lower-case alphabet g. Thus, the Lie algebra of $SO(n)$ is denoted by $so(n)$ and that of $SU(n)$ by $su(n)$.

The BCH formula can then be used to define an operation $\star$ involving to elements A, B in the Lie Algebra

$$e^A e^B = e^{A \star B} \ . \tag{1}$$

Thus the operation $\star$ is given by

$$A \star B \equiv A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] - \frac{1}{12}[B,[A,B]] + \cdots \tag{2}$$

where $[A,B]$ is the Lie bracket.

The associativity of group multiplication is equivalent to the Lie brackets satisfying the Jacobi identity. The simplest way to see this is to consider the equality to third order in λ by expanding both sides using the BCH formula twice.

$$\left(e^{\lambda A} \cdot e^{\lambda B}\right) \cdot e^{\lambda C} = e^{\lambda A} \cdot \left(e^{\lambda B} \cdot e^{\lambda C}\right) \tag{3}$$

Exercise: Verify equality of the above expression up to second order.

We now consider terms at third-order in λ, The third-order term of the LHS is

$$\begin{aligned} \text{LHS}|_3 &= \frac{1}{4}[[A,B],C] + \frac{1}{12}[A,[B,C]] + \frac{1}{12}[B,[A,C]] \\ &+ \frac{1}{12}[A,[A,C] + \frac{1}{12}[B,[B,C]] - \frac{1}{12}[C,[A,C]] - \frac{1}{12}[C,[B,C]] + \frac{1}{12}[A,[A,B] - \frac{1}{12}[B,[A,B]] \end{aligned}$$

The third-order term of the RHS is

$$\text{RHS}|_3 = \frac{1}{4}[A,[B,C]] - \frac{1}{12}[B,[A,C]] - \frac{1}{12}[C,[A,B]]$$
$$+\frac{1}{12}[A,[A,B] + \frac{1}{12}[A,[A,C]] - \frac{1}{12}[B,[A,B]] - \frac{1}{12}[C,[A,C]] + \frac{1}{12}[B,[B,C] - \frac{1}{12}[C,[B,C]]$$

Notice that the second line of both equations are identical. Thus the condition that the two third-order terms coincide is equivalent to the equality of the first lines of the above two equations. This is equivalent to

$$\frac{1}{6}\left([A,[B,C]] + [B,[C,A]] + [C,[A,B]]\right) = 0 \ , \tag{4}$$

which is the Jacobi identity for the Lie bracket. Thus, the associativity of the group multiplication implies that the Lie bracket must satisfy the Jacobi identity.

Exercise: A *symplectic matrix* S is a real matrix that satisfies

$$S \cdot \Omega \cdot S^{\mathrm{T}} = \Omega \ ,$$

where $\Omega = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$. Show that $\det(S) = +1$.

Any symplectic matrix can be written as $S = e^B$. Show that

(i) $B \cdot \Omega = -\Omega \cdot B^{\mathrm{T}}$ or $(B\Omega) = (B\Omega)^{\mathrm{T}}$.

(ii) $\mathrm{Tr}(B) = 0$,

(iii) Let B_1 and B_2 be two matrices that satisfy the condition given in part (i). Then, $[B_1, B_2]$ also satisfies the same condition ('closure').

The set of matrices B that satisfy condition (i) form the Lie algebra $sp(2n)$ which gives the Lie group $Sp(2n)$ on exponentiation.