DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH3500 Classical Physics **Problem Set 10** **19.11.2009**

1. Show that the grand partition function for a quantum ideal gas with chemical potential μ in a volume V at temperature T is given by
$$\log Q(V,T,\mu) = -\eta \frac{2\pi g V(2m)^{3/2}}{h^3} \int_0^\infty d\varepsilon\ \varepsilon^{1/2}\ \log\left(1-\eta e^{\beta(\mu-\varepsilon)}\right)\ ,$$
where $\eta = +1$ for bosons and -1 for fermions; $g = (2s+1)$ is the internal degeneracy due to the spin of the particle. Note that $\eta = \exp(2\pi i s)$ as a consequence of the spin-statistics theorem.

2. By computing the pressure, show that the equation of state for the quantum gas can be written as
$$\frac{PV}{k_B T} = \log Q(V,T,\mu)\ .$$

3. By integrating by parts, show that one can rewrite the grand partition function and hence the grand partition function can be written as
$$\Phi(V,T,\mu) = -\frac{4\pi g V(2m)^{3/2}}{3h^3} \int_0^\infty \frac{d\varepsilon\ \varepsilon^{3/2}}{e^{\beta(\varepsilon-\mu)}-\eta}\ .$$
(Verify that the surface term obtained on integrating by parts vanishes.)

4. Using the above result, show that
$$\begin{aligned} U \equiv \langle E\rangle &= \frac{2\pi g V(2m)^{3/2}}{h^3} \int_0^\infty \frac{d\varepsilon\ \varepsilon^{3/2}}{e^{\beta(\varepsilon-\mu)}-\eta}\ , \\ P &= -\frac{\partial\Phi}{\partial V} = \frac{4\pi g(2m)^{3/2}}{3h^3} \int_0^\infty \frac{d\varepsilon\ \varepsilon^{3/2}}{e^{\beta(\varepsilon-\mu)}-\eta}\ . \end{aligned}$$
It is easy to see that $P = \frac{2}{3}\frac{U}{V} = \frac{2}{3}u$, thus proving the desired relation.

5. Using $\langle N\rangle = -\frac{\partial\Phi}{\partial\mu}$, show that ($\lambda \equiv \frac{h}{(2\pi m k_B T)^{1/2}}$)
$$\frac{3\sqrt{\pi} n\lambda^3}{4g} = \int_0^\infty \frac{dx\ x^{3/2} e^{(x-\beta\mu)}}{(e^{(x-\beta\mu)}-\eta)^2}\ .$$

6. Consider the function $g(\varepsilon) = \frac{\beta e^{\beta(\varepsilon-\mu)}}{(e^{\beta(\varepsilon-\mu)}+1)^2}$ – this is the derivative of the Fermi function. Show that it is sharply peaked about $\varepsilon = \mu$ at low temperature and can be replaced by $\delta(\varepsilon-\mu)$ at zero temperature. By studying $\log g(\varepsilon)$, argue that one can carry out the **approximation**:
$$g(\varepsilon) \overset{T\to 0}{\longrightarrow} \left[\frac{\beta}{2\sqrt{\pi}}\ e^{-\beta^2(\varepsilon-\mu)^2/4}\right] \overset{T=0}{\longrightarrow} \delta(\varepsilon-\mu)\ .$$
Hence, show that for a Fermi gas, one has
$$\varepsilon_F^{3/2} \equiv \frac{3nh^3}{4\pi(2m)^{3/2}} \sim \int_{-\infty}^\infty d\varepsilon \varepsilon^{3/2} \frac{\beta}{2\sqrt{\pi}}\ e^{-\beta^2(\varepsilon-\mu)^2/4}\ .$$

7. By expanding $\varepsilon^{3/2}$ about μ to second order, one can carry out the Gaussian integrations to obtain
$$\varepsilon_F^{3/2} = \mu^{3/2} + \frac{3k_B^2 T^2}{4\mu^{1/2}} + \mathcal{O}(T^3)\ .$$
Hence, one sees that the chemical potential reduces as temperature increases.
$$\mu(n,T) = \varepsilon_F - \gamma\ (k_B^2 T^2/\varepsilon_F) + \mathcal{O}(T^3)\ ,$$
where γ is a positive constant. Determine γ. (Compare with the exact answer: $\gamma = \pi^2/8$)