DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH350 Classical Physics **Problem Set 2** **14.8.2009**

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1. We derive the lens-maker's formula as an application of Fermat's principle.

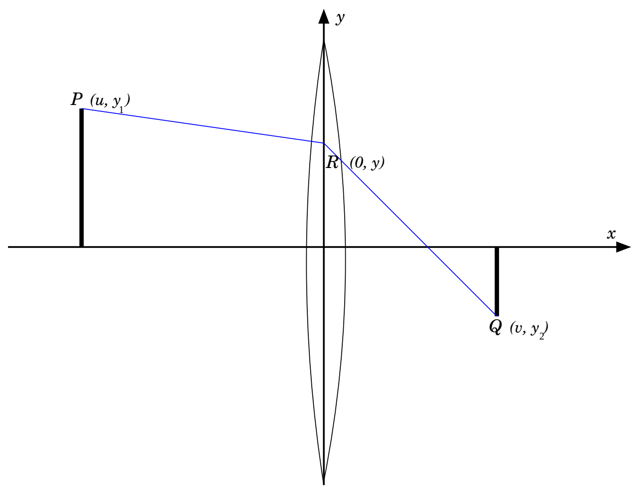

Figure 1: Deriving the lens maker's formula

   (a) Consider a ray beginning at P (with coordinates (u, y_1)) and ending at Q (with coordinates (v, y_2). Consider the family of trajectories parametrized by the variable y as shown in the figure. Obtain the function $T(y)$ that represents the time taken by the trajectory passing through $(0, y)$ in the lens after taking into account the additional delay $D(y) = D(-y)$ due to the lens. Using Fermat's principle, obtain the trajectory of the light ray assuming that $|u| \gg y_1$ and $|v| \gg y_2$ and keeping terms up to and including order y.

   (b) For an image to be formed at Q, all trajectories *must* reach the point Q with the same delay. For this to happen, the terms in $T(y)$ independent of y and proportional to y should separately vanish. Show that you recover the standard lens law with $f = -1/cD''(0)$, where c is the speed of light and $D(y) = D(0) + D''(0)\ \frac{y^2}{2} + \ldots$.

   (c) Consider a thin bi-convex lens made of two faces with radii of curvature R_1 and R_2 and the index of refraction of the lens is taken to be n. Compute the delay and hence obtain the lens-maker's formula for a biconvex lens.

2. Obtain the action functional for a particle of mass m moving in three-dimensions under the influence of a potential $V(x, y, z)$ in three different coordinate systems: Cartesian, cylindrical polar and spherical polar coordinates. Then, derive the equations of motion in each of these coordinate systems.

3. **The Brachistochrone problem:** Jacob Bernoulli in 1696 posed the following problem. Consider two points, $A = (0,0)$ and $B(x_0, y_0)$ such that $y_0 < 0$ with gravity acting along the negative y-axis. Find a curve connecting A to B (or equivalently, construct a frictionless slide connecting A to B) such that a particle which starting at rest from A and falling under the influence of gravity takes the least time. Construct the time functional for the particle and obtain and solve the Euler-Lagrange equations of motion. The curve that you obtain is a cycloid (it is also called the Brachistochrone curve in Greek: *Brakhus*=shortness and *chronos*=time.) Do take a look at the following URL only after you have completely solved the problem!
`http://www.mathcurve.com/courbes2d/brachistochrone/brachistochrone.shtml`

4. A free particle of unit mass moving in one dimension with coordinate q. We are given that the particle was $q = 1$ at $t = 0$ and at $q = 5$ at time $t = 1$. The action for this particle is

$$S[q(t)] = \frac{1}{2}\int_0^1 dt\,(\dot{q})^2 \ .$$

Consider the following four possible trajectories of the particles:

$$\begin{aligned} q_A(t) &= 2\sin(2\pi t) + 4t + 1 \ , \\ q_B(t) &= -3t^2 + 7t + 1 \ , \\ q_C(t) &= 4t + 1 \ , \\ q_D(t) &= -2\cos(\pi t) + 3 \ . \end{aligned}$$

Of the four paths, $q_C(t)$ is the solution to the equations of motion – this is usually called the *classical solution*. Compute the action for all four trajectories and verify that the minimum occurs for $q_C(t)$.

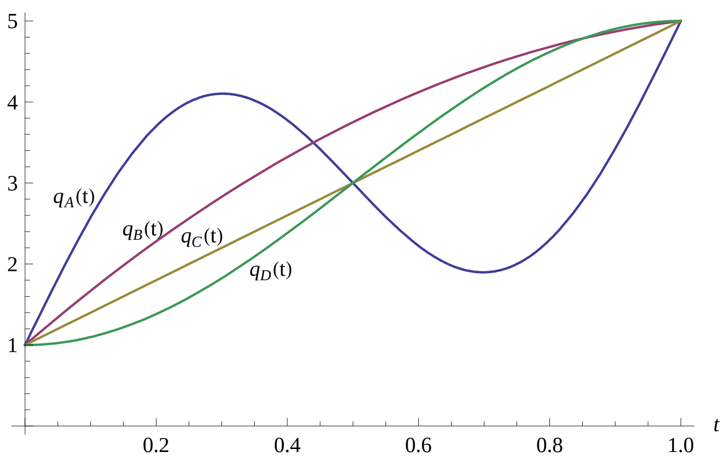

Figure 2: A plot of the four trajectories. Note that all paths begin at $q = 1$ and end at $q = 5$.

5. Consider the action for a particle of unit mass in a harmonic potential

$$S[q(t)] = \frac{1}{2}\int_0^1 dt \left[(\dot{q})^2 - \pi^2(q-3)^2\right] \ .$$

Evaluate the action for the four trajectories that were considered in the previous problem. Show that the minimum value occurs for the solution $q_D(t)$ and verify that it is indeed the classical solution.
