PH350 Classical Physics **Problem Set 3** **27.8.2009**

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1. A simple pendulum consists of a rigid massless rod of fixed length l with a bob of mass m attached to its end. Its potential energy when its angular displacement from the vertical is θ is given by $mgl(1-\cos\theta)$.

   (a) Write down the Hamiltonian of the system, and the corresponding Hamilton's equations of motion. What does the latter reduce to in the small-θ approximation? Note that the generalised momentum $p_\theta = ml^2\dot{\theta}$.

   (b) Identify the phase space for the above system, and obtain the equation describing the phase trajectories of the system. Note that there are two types of motion possible for the system – oscillation and rotation, respectively (the rod is rigid). Sketch the phase potrait (the family of phase trajectories), including the separatrices that demarcate the transition from oscillatory to rotational motion. What is the equation describing these separatrices?

   (c) Obtain a formal expression for the time period of the above system in the case of (i) oscillatory motion, (ii) rotational motion. What does this reduce to in the case of small oscillations about the vertical? What is the time period for motion on a separatrix?

   (d) Consider the case when the total energy is exactly equal to that on a separatrix. Given that at $t=0$, the bob is located at $\theta=0$ and is moving with a positive velocity, solve Hamilton's equations to obtain $\theta(t)$ and $p_\theta(t)$ explicitly as functions of time. Schematically sketch both quantities vs time. Extend each sketch to include times $(-\infty, 0)$.

2. A particle of mass m has the Hamiltonian (in spherical polar coordinates)

$$H(r,\theta,\varphi,p_r,p_\theta,p_\varphi) = \frac{{p_r}^2}{2m} + \frac{{p_\theta}^2}{2mr^2} + \frac{{p_\varphi}^2}{2mr^2\sin^2\theta} - \frac{K}{r} \quad ,$$

   where K is a positive constant.

   (a) Write down Hamilton's equations of motion for the particle.

   (b) Express p_r, p_θ and p_φ in terms of the corresponding velocities $\dot{r}$, $\dot{\theta}$ and $\dot{\varphi}$ and the coordinates r, θ and φ.

   (c) Is there any coordinate on which H does *not* explicitly depend? What does this imply for the corresponding conjugate momentum?

   (d) Determine whether ${p_\theta}^2 + \frac{{p_\varphi}^2}{\sin^2\theta}$ is a constant of motion. (There is a *very* easy way to do this!)

(e) Express the momentum $\vec{p}$ in spherical polar coordinates (i.e., expand it in terms of the unit vectors $\hat{e}_r$, $\hat{e}_\theta$ and $\hat{e}_\varphi$). Hence find $\vec{L} = \vec{r} \times \vec{p}$, and thence $\vec{L} \cdot \vec{L}$, in spherical polar coordinates. Do you recognize the result?

3. The Hamiltonian for a system with one degree of freedom has the form

$$H = \frac{p^2}{2a} - bqpe^{-\alpha t} + \frac{ba}{2} q^2 e^{-\alpha t}(\alpha + be^{-\alpha t}) + \frac{kq^2}{2} \quad ,$$

where a, b, α, and k are constants.

   (a) Find a Lagrangian corresponding to this Hamiltonian.

   (b) Find an equivalent Lagrangian that is not explicitly dependent on time.

   (c) What is the Hamiltonian corresponding to the second Lagrangian, and what is its relationship between the two Hamiltonians?

4. A particle of mass m is restricted to move on the surface of a cylinder of radius R. The z-axis is the axis of the cylinder. Further, the particle is bound to the origin by a spring (with a spring constant K and unstretched length$= 0$).

   (a) Show that the Lagrangian for the particle is given by (φ is the polar angle)

$$L = \frac{m}{2}\left(R^2\dot{\varphi} + \dot{z}^2\right) - \frac{K}{2}\left(R^2 + z^2\right) \ .$$

   (b) Obtain the generalised momenta conjugate to the coordinates φ and z.

   (c) Obtain the Hamiltonian and write down the Hamilton's equations of motion.

   (d) Identify the different kinds of motion that are possible.

5. The point of suspension of a simple pendulum of length l and mass m is constrained to move on a parabola $z = ax^2$ in the vertical $x - z$ plane. Derive a Hamiltonian governing the motion of the pendulum and its point of suspension. Obtain the Hamilton's equations of motion.
