DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH350 Classical Physics **Problem Set 5** **17.9.2009**

1. A pair of (distinguishable) dice is tossed once. Each die can give a score of 1, 2, 3, 4, 5, or 6. Let s denote the total score of the pair of dice.

   (a) What is the most probable value of s?

   (b) Find the mean value of s, the mean square value of s, and the standard deviation of s.

2. A rod of unit length is broken into three pieces at random. (More precisely, the two points where the cuts are made are chosen independently, and with a uniform probability all along the rod.) What is the probability that the three pieces can form a triangle?

3. The *Poisson distribution* is defined as

$$p(n) = \frac{e^{-\lambda}\lambda^n}{n!} \quad (n = 0, 1, 2, \cdots \text{ ad inf.}) \quad ,$$

   where λ is a positive constant. As you know it represents the average value of n, i.e., $\langle n \rangle = \lambda$. Using the generating function of $p(n)$, i.e., $f(z) = \sum_{n=0}^{\infty} p(n) z^n$, find the following quantities:

   (a) $K_2 = \langle n^2 \rangle - \langle n \rangle^2 \quad ,$

   (b) $K_3 = \langle n^3 \rangle - 3\langle n^2 \rangle \langle n \rangle + 2\langle n \rangle^3 \quad ,$

   (c) $K_4 = \langle n^4 \rangle - 4\langle n^3 \rangle \langle n \rangle - 3\langle n^2 \rangle^2 + 12\langle n^2 \rangle \langle n \rangle^2 - 6\langle n \rangle^4 \quad .$

   The quantity K_n is called the n^{th} *cumulant* of the distribution.

4. Repeat Prob. 3 for the *geometric distribution*

$$p(n) = \frac{1}{\lambda + 1} \left(\frac{\lambda}{\lambda + 1} \right)^n \quad (n = 0, 1, 2, \cdots \text{ ad inf.}) \quad ,$$

   where λ is a positive constant.

5. For *continuous* random variables, one speaks of a *probability density function* (p.d.f.) rather than a probability distribution as in the case of discrete random variables. Thus if a random variable can take all values in $(-\infty, \infty)$, it has a p.d.f. $p(x)$ such that:

   (i) $p(x)dx$ is the probability that the random variable has a value in the infinitesimal *range* dx about the value x.

(ii) $p(x) \geq 0$ for all x .

(iii) $\int_{-\infty}^{\infty} p(x)dx = 1$.

Sometimes the density is loosely called a "distribution", but what is meant should be clear from the context. The *Gaussian* or *normal distribution* with mean μ and variance σ^2 is defined by the p.d.f.

$$p(x) = (2\pi\sigma^2)^{-1/2} \ \exp\left[-(x-\mu)^2/2\sigma^2\right] \quad , \quad -\infty < x < \infty \quad .$$

(a) Check that

$$\begin{aligned} \int_{-\infty}^{\infty} p(x)dx &= 1 \quad \text{(normalization)} \\ \langle x \rangle = \int_{-\infty}^{\infty} x \ p(x)dx &= \mu \quad \text{(mean)} \\ \langle x^2 \rangle - \langle x \rangle^2 &= \sigma^2 \quad \text{(variance).} \end{aligned}$$

(b) The quantity $F(x) = \int_{-\infty}^{x} p(x)dx$ is the total probability that the random variable has a value $\leq x$. It is called the *cumulative distribution function* (c.d.f.) or distribution function in brief. Sketch $F(x)$ versus x for the Gaussian case.

6. Let x and y be two independent random variables. Each has a Gaussian p.d.f. with zero mean, and the variances are σ_1^2 and σ_2^2 respectively. Show that the random variable $z = x + y$ also has a Gaussian p.d.f., with zero mean and variance $\sigma^2 = \sigma_1^2 + \sigma_2^2$. This is a very important property of the normal distribution.

7. A random variable x has a *uniform* p.d.f. in the range $0 \leq x \leq 1$. In other words,

$$p(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Another, independent, random variable y also has a uniform p.d.f. in the range $0 \leq y \leq 1$. Find the p.d.f. of the *sum* of x and y, i.e., of the random variable $z = x + y$. Sketch the p.d.f. of z as a function of z.

8. A random variable x with zero mean has the Gaussian p.d.f. $p(x) = (2\pi\sigma^2)^{-1/2} \ \exp\left(-x^2/2\sigma^2\right)$. Find the p.d.f. of the variable $\xi = x^2$. This has an immediate physical application. According to the Maxwellian distribution of velocities of the molecules of a gas, each component of the velocity of a molecule has a Gaussian p.d.f. - e.g.,

$$p(v_x) = \left(\frac{m}{2\pi k_B T}\right)^{\frac{1}{2}} \exp\left(-m v_x^2/2k_B T\right) \quad (-\infty < x < \infty) \quad ,$$

and similarly for v_y and v_z. (Here T is the absolute temperature and m is the mass of a molecule.) Use this to find the p.d.f. of the energy ϵ of the particle, where $\epsilon = m(v_x^2 + v_y^2 + v_z^2)/2$.