DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH3500 Classical Physics **Problem Set 9** **16.11.2009**

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1. We have seen in class that the principle of relativity implies that the electric and magnetic fields must mix. In particular, one has the transformation rules for **E** and **B** under a boost from a frame S to a frame S', which is moving with an arbitrary velocity **u** ($u < c$) as seen from S. Let unprimed and primed quantities denote variables in S and S' respectively. Further, let the subscripts $\parallel$ and $\perp$ denote components respectively along the direction of the boost **u** and perpendicular to it. Then:

$$E'_{\parallel} = E_{\parallel}\,, \quad \mathbf{E}'_{\perp} = \gamma\left(\mathbf{E}_{\perp} + \mathbf{u}\times\mathbf{B}_{\perp}\right)$$
$$B'_{\parallel} = B_{\parallel}\,, \quad \mathbf{B}'_{\perp} = \gamma\left(\mathbf{B}_{\perp} - \frac{\mathbf{u}\times\mathbf{E}_{\perp}}{c^2}\right)\,.$$

where $\gamma = (1 - u^2/c^2)^{-1/2}$ and the arguments of the fields are taken be in the spacetime coordinates of the frame.

(a) Now consider a boost along the x-axis with speed u: $x^{\mu\prime} = \Lambda^{\mu}{}_{\nu} x^{\nu}$. Explicitly write out the boost matrix $\Lambda^{\mu}{}_{\nu}$ corresponding to this boost.

(b) Let $F^{\mu\nu} = -F^{\nu\mu}$ be a second-rank antisymmetric tensor of the Lorentz group. In other words, it transforms as

$$F^{\mu\nu\prime} = \Lambda^{\mu}{}_{\rho}\ \Lambda^{\nu}{}_{\sigma}\ F^{\rho\sigma}\ .$$

Verify that on making the identifications in Gaussian units, $F^{0i} = -E_i$ and $F^{ij} = -\epsilon_{ijk} B_k$, the transformation of the second-rank tensor under the above boost is the same as the one given above. In particular, see that F^{01} and F^{12} are invariant under the boost matrix given in part (a) above and the other components mix as required.

(c) Using the above identifications, show that the **electromagnetic field strength** $F^{\mu\nu}$ can be written as

$$F^{\mu\nu} \equiv \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}\ ,$$

where $A^{\mu} = (\phi, \mathbf{A})$ is the four-vector constructed from the scalar and vector potentials.

(d) Verify that $(E^2 - B^2)$ and $\mathbf{E}\cdot\mathbf{B}$ transform as Lorentz scalars. Obtain expressions for them in terms of the electromagnetic field strength as well as invariant tensors such as $\eta_{\mu\nu}$ and the Levi-Civita tensor $\epsilon_{\mu\nu\rho\sigma}$.

2. (a) Show that the conservation of four-momentum makes it is impossible for an electron and a positron to annihilate and produce a single photon. (It is however possible for it to decay into two photons.)

   (b) A particle of rest mass m has three-velocity $\mathbf{v}$. Determine it energy up to order v^4. What is speed at which the fourth-order term is equal to half of the second-order term, $\frac{1}{2}mv^2$.

3. A neutron at rest decays into a proton, electron and an electron anti-neutrino. ($m_n = 939.5654$MeV, $m_p = 938.2720$MeV and $m_e = 0.5110$MeV)

$$n \longrightarrow p^+ + e^- + \bar{\nu}_e \ .$$

   (a) For the moment, neglect the anti-neutrino and estimate the four-momentum of the electron and proton using the conservation of total four-momentum of the system. Is the electron relativistic? What about the proton?

   (b) Assuming that the anti-neutrino has mass m_ν, now rework the implication of the conservation of total four-momentum. Note that the three-momenta of the proton, electron and the anti-neutrino must necessarily lie in a plane. Why? Solve for the (magnitude of the) three-momentum of the anti-neutrino in terms of the electron's three-momentum.