DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH5460 Classical Field Theory   Assignment 3   25.8.2010 (due: 30.8.2010)

(Mostly) Finite Groups

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1. The dihedral group, D_{2n}, is generated by two elements x and y satisfying the relations
$$x^n = 1, \quad y^2 = 1, \quad \text{and} \quad yx = x^{n-1}y \ . \qquad (*)$$
   (a) By working out the multiplication table for D_3, prove the isomorphism, $D_6 \sim S_3$.
   (b) Obtain subgroups, with order 2 and 3, of S_3 and check to see if they are *normal* subgroups. If they are normal, obtain the group multiplication law for the corresponding coset. Is S_3 a simple group?

2. Consider the cyclic group $\mathbb{Z}_3 = (e, g, g^2)$ with $g^3 = e$. The regular representation is a map from the group to 3×3 matrices with g being represented by the matrix
$$g \longrightarrow M(g) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \ ,$$
with the property that $M(g_1)M(g_2) = M(g_1 \cdot g_2)$ for any $g_1, g_2 \in \mathbb{Z}_3$.
   (a) What does the identity $g^3 = e$ become in this representation? Verify that it holds.
   (b) Obtain the similarity transformation that diagonalises $M(g)$.

3. Obtain the condition under which the 2×2 matrix U given below is an $SU(2)$ matrix.
$$U = \alpha_0 I + i\alpha_i \sigma^i \quad ,$$
where σ^i are the Pauli sigma matrices and α_0, α_i are real parameters. What is the geometric interpretation of this condition?

4. Since the Pauli sigma matrices form a basis for traceless 2×2 hermitian matrices, an arbitrary $SU(2)$ matrix (in the two dimensional representation) can also be written as
$$U^{(2)}(\hat{n}, \theta) = \exp\left(i\frac{\theta}{2}\hat{n} \cdot \vec{\sigma}\right) \quad ,$$
where $|\hat{n}| = 1$. Solve for α_0, α_i in terms of θ and $\hat{n}$. Hence show that $U(\hat{n}, 2\pi) = -I$.

5. **This question will not be graded.** List out all finite groups of order ≤ 10 and identify all the simple groups in that list.