DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH5460 Classical Field Theory — Assignment 4 — 30.8.2010

Functional Calculus and Equations of Motion

1. Consider the following functionals of N scalar fields $\phi^i(x)$ (m and λ are constants)

$$
\begin{aligned}
T_0 &= \int d^4x \, \sum_{a=1}^{N} \eta^{\mu\nu}(\partial_\mu \phi_a)(\partial_\nu \phi_a) \quad , \\
U_0 &= \int d^4x \left(m^2 \sum_{a=1}^{N} (\phi_a)^2 + \lambda \left[\sum_{a=1}^{N} (\phi_a)^2 \right]^2 \right) \quad ,
\end{aligned}
$$

where $\mu = 0, 1, 2, 3$, $\eta^{\mu\nu} = \text{Diag}(1, -1, -1, -1)$ and $\partial_\mu = \partial/\partial x^\mu$. Evaluate the following:

$$
(a) \ \frac{\delta \phi_a(x)}{\delta \phi_b(y)} \quad \text{and} \quad (b) \ \frac{\delta \partial_\mu \phi_a(x)}{\delta \phi_b(y)}
$$

$$
(c) \ \frac{\delta T_0}{\delta \phi_b(y)} \quad \text{and} \quad (d) \ \frac{\delta U_0}{\delta \phi_b(y)}
$$

Using the result of the last two parts, obtain the equations of motion for the action: $S_0 = T_0 - U_0$.

2. Consider the following functionals of a vector field $A_\mu(x)$ (m is a constant of suitable dimensions)

$$
\begin{aligned}
T_1 &= -\frac{1}{4} \int d^4x \ F_{\mu\nu} F_{\rho\sigma} \eta^{\mu\rho} \eta^{\nu\sigma} \quad , \\
U_1 &= m^2 \int d^4x \ A_\mu A_\nu \eta^{\mu\nu} + 4\pi \int d^4x \ J^\mu A_\mu \quad ,
\end{aligned}
$$

where $F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\nu)$ and J^μ is a four-current. Evaluate the following:

$$
(a) \ \frac{\delta F_{\mu\nu}(x)}{\delta A_\rho(y)} \qquad (b) \ \frac{\delta T_1}{\delta A_\rho(y)} \qquad (c) \ \frac{\delta U_1}{\delta A_\rho(y)}
$$

Using the result of the last two parts, obtain the equations of motion for the *Proca action*: $S_1 = T_1 - U_1$. Do you recognise the equations of motion in the $m = 0$ limit?