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PH5460 Classical Field Theory Assignment 7 16.9.2010 (due: 27.9.2010)

Lie Algebras

It is conventional to indicate the Lie algebra associated with a group G by g, the corresponding letter in lower case. Thus the Lie algebra of the Lie group $SU(n)$ is written as $su(n)$ and so on.

1. A basis for the Lie Algebra of $so(N)$ is given by the following $N \times N$ matrices:
$$(M_{mn})_{ab} \equiv \delta_{ma}\delta_{nb} - \delta_{mb}\delta_{na} \ .$$
Note that all indices m, n, a, b run from 1 to N.
   (a) Show that the commutator (Lie Bracket) of the basis matrices is given by
$$[M_{mn}, M_{pq}] = \delta_{np}M_{mq} - \delta_{mp}M_{nq} - \delta_{nq}M_{mp} + \delta_{mq}M_{nn} \ .$$
   (b) Show that when $N = 3$, the (above) Lie algebra is isomorphic to the $su(2)$ Lie algebra.
   (c) Show that when $N = 4$, the Lie algebra is isomorphic to $su(2) \oplus su(2)$.
   Hint: Consider the two sets of linear combinations $(M_{ab} \pm \alpha\epsilon_{abc}M_{c4})$ for $a, b, c = 1, 2, 3$ for a suitable chosen α.
   (d) Show that dimensions of the Lie algebras $so(6)$ and $su(4)$ are the same. This is a necessary but not sufficient condition for $so(6) \sim su(4)$.

2. **The $su(3)$ Lie algebra** The $su(3)$ Lie algebra is generated by the LVS of 3×3 antisymmetric matrices. A standard choice for this basis is given by the (eight) Gellmann λ-matrices:
$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \ \lambda_2 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \ \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \ \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \ \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} , \ \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

(a) Verify that $\mathrm{Tr}(\lambda_a \lambda_b) = 2\delta_{ab}$.

(b) The generators of $su(3)$ are chosen to be $T_a = \frac{1}{2}\lambda_a$. The Cartan subalgebra is chosen to be $H = (h_1 = T_3, h_2 = T_8)$. Let Ad_{h_i} $(i = 1, 2)$ be the linear maps from the Lie algebra to itself defined as follows:

$$\mathrm{Ad}_{h_i} : \ x \mapsto [h_i, x] \quad \forall x \in su(3) \ .$$

Organize the remaining six generators such that they are simulataneous eigenvectors under the two maps i.e., $[h_i, x] = \alpha_i x$ for $i = 1, 2$.

(c) Hence, obtain the Cartan decomposition of $su(3) = \mathcal{L}^+ \oplus H \oplus \mathcal{L}^-$. Note that $\mathcal{L}^\pm$ are both subalgebras of dimension three.

(d) The structure constants are defined by the following relation:

$$[T_a, T_b] = i f_{ab}{}^{c}\, T_c \ .$$

Obtain the structure constants and show that f_{abc} is totally antisymmetric.

Recommended Reading: Equivalences of Lie Algebras:
`http://sgovindarajan.wikidot.com/equivalence-lie-algebras`