

**DEPARTMENT OF PHYSICS**  
**INDIAN INSTITUTE OF TECHNOLOGY, MADRAS**

PH5460 Classical Field Theory    Assignment 8    16.9.2010 (due: 30.9.2010)

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**Monopoles and the Dirac quantization condition**

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In this problem set, we will study a new class of solitonic solutions that appear in  $3 + 1$  dimensions. We begin first by considering Maxwell's equations in vacuum which admit the following **duality** symmetry:

$$\mathbf{E} \rightarrow \mathbf{B} \quad , \quad \mathbf{B} \rightarrow -\mathbf{E} .$$

The addition of source terms to Maxwell's equations breaks this symmetry since there are no objects that carry magnetic charge. In particular, one has  $\nabla \cdot \mathbf{B} = 0$ . Let us assume for a moment that this equation is violated at one point, say the origin, by the presence of a point magnetic monopole with magnetic charge  $g$ . In analogy with a point electric charge, the magnetic field due to a monopole is given by

$$\mathbf{B}_M = g \frac{\mathbf{r}}{r^3} .$$

It follows that  $\nabla \cdot \mathbf{B} = 4\pi g \delta^3(\mathbf{r})$ . This violates Maxwell's equations only at the location of the point monopole.

At all points in space other than the origin,  $\mathbb{R}^3 - \{0\}$ , one still has  $\nabla \cdot \mathbf{B} = 0$  and one can try to solve for  $\mathbf{B}$  in terms of a vector potential  $\mathbf{A}$ . As we will demonstrate below, this vector potential **cannot** be valid at all points in space even after deleting the origin. It is natural to assume that there exists a vector potential  $\mathbf{A}$ , such that  $\nabla \times \mathbf{A} = \mathbf{B}_M$ . Such a vector potential  $\mathbf{A}$  **must** necessarily fail to give the correct magnetic field on a set of points,  $\mathcal{S}$ , that form a string (of magnetic flux) connecting the origin to spatial infinity. This string is called the **Dirac string**. Thus, the vector potential is valid **only** on points not including  $\mathcal{S}$ . Below we discuss two such potentials,  $\mathbf{A}^N$  and  $\mathbf{A}^S$ , that have as Dirac strings along the positive and negative  $z$ -axis respectively.

As it stands, the Dirac monopole may seem to be a mathematical curiosity. We will see later on in this course that this singular solution of Maxwell's equations can be embedded into a system with non-abelian gauge symmetry as a non-singular and smooth solution. The first such (approximate) solution was exhibited independently by Polyakov and 't Hooft and is referred to as the *'t Hooft-Polyakov monopole*. In a later assignment, you will study a simplified and exact solution due to Prasad and Sommerfield that carries both electric and magnetic charge – such solutions are called **dyons**.

The quantization of electric charge as shown by the Millikan's oil drop experiment remained an unexplained result till Dirac showed in 1948 that the existence of a magnetic monopole (with magnetic charge  $g$ ) implied the quantization of electric charge. He considered a particle of electric charge  $e$  in the presence of such a Dirac monopole. A simple quantum mechanical argument

(not discussed here) that such a particle should not see the Dirac string (which is after all an artefact of working with a vector potential) implies that

$$e \cdot g = \frac{1}{2}n\hbar c \quad \text{for some integer } n .$$

We now quote Dirac: *Thus the mere existence of one pole of strength  $g$  would require all electric charges to be quantized in units of  $\hbar c/2g$  and, similarly, the existence of one charge would require all poles to be quantized.* This is now known as the **Dirac quantization condition**.

1. Show that the following vector potential

$$\mathbf{A}^S = -\frac{gy}{r(r+z)}\hat{e}_x + \frac{gx}{r(r+z)}\hat{e}_y \quad ,$$

leads to a magnetic field

$$\mathbf{B}^S = g\frac{\mathbf{r}}{r^3} + 4\pi g\delta(x)\delta(y)\theta(-z)\hat{e}_z \quad ,$$

where  $\theta(x) = 1$  for  $x > 0$  and zero otherwise. It is easy to see that  $\mathbf{B}_M = \mathbf{B}^S$  on all points except on the negative  $z$ -axis. Rewrite the vector potential in spherical polar coordinates.

2. Similarly show that the following vector potential  $\mathbf{A}^N$

$$\mathbf{A}^N = \frac{gy}{r(r-z)}\hat{e}_x - \frac{gx}{r(r-z)}\hat{e}_y \quad ,$$

leads to a magnetic field which agrees with  $\mathbf{B}_M$  on all points except on the positive  $z$ -axis. Rewrite the vector potential in spherical polar coordinates.

3. Further, show that  $\mathbf{A}^N$  and  $\mathbf{A}^S$  are related by a gauge transformation which is singular on the  $z$ -axis. As is clear from the two vector potentials  $\mathbf{A}^N$  and  $\mathbf{A}^S$ , there is always a line (the Dirac string) from the monopole to infinity where the magnetic field of the vector potential *disagrees* with that of the monopole. However, the location of the string is not *gauge invariant*. For the two vector potentials that we constructed, the Dirac string changes from the positive  $z$ -axis to the negative  $z$ -axis.
4. One may think that it is possible to find a vector field  $\mathbf{A}$  which provides a magnetic field identical to that of the magnetic monopole at all points away from the true singularity at the origin. Show that if this is true, then the flux through any closed surface enclosing the origin must be zero.

### References:

1. P.A.M. Dirac, Quantised Singularities in the Electromagnetic Field, Proc. Roy. Soc. **A133**, 60 (1931).
2. P.A.M. Dirac, *The theory of magnetic poles*, Phy. Rev. **74**, 817 (1948).