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PH3520 Quantum Physics **Problem Set 1** **6.1.2011**

Linear vector spaces (Finite dimensional case)

1. Let M be the $2n \times 2n$ dimensional m atrix
$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \ ,$$
where A, B, C, D are $n \times n$ matrices. Express its determinant in several ways specifying your assumptions. When is $\det(M) = \det(AD) - \det(BC)$?

2. Consider the 4×4 cyclic shift matrix
$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$
Show that P is a normal matrix and find its eigenvalues and corresponding eigenvectors. Hence, find the matrix that diagonalises P.

3. Check whether the following sets of elements form an LVS. If they do, find the dimensionality of the LVS.
   (a) The set of all $n \times n$ matrices with complex entries.
   (b) The set of all polynomials (of order $\leq n$) of a complex variable z.
   (c) The set of all tensors of rank 2 in three dimensions.
   (d) The set of all antisymmetric tensors of rank 2 in three dimensions.
   (e) The set of all 2×2 matrices whose trace is zero.
   (f) The set of all solutions of the differential equation $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 0$.
   (g) The set of all $n \times n$ unitary matrices. (U is unitary iff $U^\dagger U = UU^\dagger = I$.)
   (h) The set of all $n \times n$ hermitian matrices (with multiplication by real scalars).

4. **The Cauchy-Schwarz inequality** is of fundamental importance. It says that $|\langle u, v\rangle| \leq ||u||\ ||v||$, for any two vectors $u,\ v \in \mathbb{V}$, the equality holding iff u and $|v$ are linearly dependent. In terms of ordinary vectors in Euclidean space, it amounts to saying that the cosine of the angle between two vectors has a magnitude between 0 and 1, the limiting value of unity occurring iff the vectors are collinear. Establish the Cauchy-Schwarz inequality. *Hint*: Consider the inner product $\langle u{+}a\,v\,,\ u{+}a\,v\rangle$ where a is an arbitrary complex number. Choosing a appropriately leads to the desired inequality.

5. Use the Cauchy-Schwarz inequality to establish the "triangle" or **Minkowski inequality** $\parallel u+v \parallel \leq \parallel u \parallel + \parallel v \parallel$ for any two vectors u and $v \in \mathbb{V}$.

6. In an n-dimensional LVS (with standard norm), consider the vectors v_k $(k = 1,\ 2, \ldots,\ n)$ defined by

$$v_1 = (1,\ 0,\ 0, \cdots,\ 0)\,, \quad v_2 = (1/\sqrt{2},\ 1/\sqrt{2},\ 0,\ 0, \cdots,\ 0)\,,$$

$$\cdots\cdots\cdots\,, \quad v_n = (1/\sqrt{n},\ 1/\sqrt{n},\ 1/\sqrt{n},\ \cdots\ ,\ 1/\sqrt{n})\,.$$

Construct a vector u such that $\langle v_k, u\rangle = 1$ for every $k\,(1 \leq k \leq n)\,.$

7. The set of all $n \times n$ matrices (with complex entries) forms an LVS (Problem 1(a) above). The inner product of two elements in this space may be defined as $\langle A,\ B\rangle = \mathrm{Tr}\,(A^\dagger B)$, where $A^\dagger$ denotes the hermitian conjugate of A. If A is an arbitrary $n \times n$ matrix, and U is an unitary $n \times n$ matrix, show that

$$\langle A,\ A\rangle\ \geq\ \frac{1}{n}\,|\langle U^\dagger,\ A\rangle|^2\,.$$

8. Let P_n be the real vector space of linear polynomials $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ of degree $\leq n$. $D \equiv d/dx$, is as a linear operator on P_n (check this).

   (a) Find the matrix of D with respect to a convenient basis, and prove that D is a nilpotent operator, i.e., $D^k = 0$ for some k.

   (b) Determine all D-invariant subspaces, i.e., subspaces of W of P_n such that $DW \subset W$.