DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH3520 Quantum Physics **Problem Set 2** **19.1.2011**

Hilbert spaces

Infinite dimensional linear vector spaces lead to some complications that are not present for a finite dimensional LVS. Two examples of infinite dimensional LVS are:

- Consider the sequence $|a\rangle = (a_1, a_2, a_3, \ldots)$. The space of such sequences $|a\rangle$ with the inner product

  $$\langle a|b\rangle \equiv \sum_i a_i^* b_i$$

  forms a LVS. ℓ^2 is the LVS consisting of all vectors $|a\rangle$ with **finite** norm.

- The space of square-integrable functions on the interval $[a, b]$ is the LVS $L^2[a, b]$. The inner product of two functions $\psi(x)$ and $\phi(x)$ is defined to be

  $$\langle\psi|\phi\rangle \equiv \int_a^b dx\ \psi^*(x)\ \phi(x)$$

Hilbert spaces are infinite dimensional linear vector spaces with some additional properties. In particular, they are *complete*, i.e., the vectors that arise as limits of Cauchy sequences are also elements of the LVS. This is similar to considerations that complete the set of rational numbers to the real line. Both the examples mentioned above are Hilbert spaces.

Let $\mathbb{V}$ denote a linear vector space, and let $\mathbb{D}$ be a subspace of $\mathbb{V}$. An **operator** A in $\mathbb{V}$ is a function which relates to each element $|\phi\rangle \in \mathbb{D}$ a particular element $A|\phi\rangle = |\psi\rangle \in \mathbb{V}$. The set Δ consisting of elements $|\psi\rangle$, as $|\phi\rangle$ runs through all the elements of $\mathbb{D}$, is called the **range** of the operator A. $\mathbb{D}$ is called the **domain** of the operator A. To avoid confusion, it is helpful to use the notation $\mathbb{D}_A$ and Δ_A for the domain and range, respectively, of any operator A in the linear space.

If A maps each pair of different elements of $\mathbb{D}_A$ into a pair of different elements of Δ_A, then A has an **inverse** A^{-1} which maps the elements of Δ_A into the elements of $\mathbb{D}_A$. We then have $A^{-1}|\psi\rangle = |\phi\rangle$ iff $A|\phi\rangle = |\psi\rangle$. Two operators are equal if they have the same domain, and if they have the same action on each given vector in their common domain.

A subset $\mathbb{U}$ of $\mathbb{V}$ is a **linear manifold** if the following property is satisfied: given any pair of elements $|\phi\rangle, |\chi\rangle \in \mathbb{U}$, any arbitrary linear combination $\alpha\,|\phi\rangle + \beta\,|\chi\rangle$ (where α and β are scalars) is also an element of $\mathbb{U}$.

A is a **linear operator** if $\mathbb{D}_A$ is a linear manifold, and if $A\,(\alpha\,|\phi\rangle + \beta\,|\chi\rangle) = \alpha\,A|\phi\rangle + \beta\,A|\chi\rangle$.

A linear operator A is **bounded** if

$$\sup_{|\phi\rangle\,\in\,\mathbb{D}_A} \frac{\|A\,\phi\|}{\|\phi\|} < \infty\,,$$

where $|\phi\rangle \in \mathbb{D}_A$. The supremum or least upper bound on the LHS in the above is called the **norm** of the operator A . If a is any scalar, then the norm of the operator $a\,A$ is given by $\|aA\| = |a|\,\|A\|$.

If A and B are linear operators in $\mathbb{V}$, then any linear combination $C = aA + bB$ (where a and b are scalars) is also a linear operator with domain $\mathbb{D}_C = \mathbb{D}_A \cap \mathbb{D}_B$. The operators AB and BA are also linear operators. If A and B are bounded linear operators in all of $\mathbb{V}$, then so are AB and BA . Further, $\|AB\| \leq \|A|\,\|B\|$.

1. Which of the following infinite sequences $(x_1, x_2, \cdots)$ listed below belong to ℓ^2 ?

   (a) $x_n = (-1)^n\,(\ln n)/n$ (b) $x_n = n!/(2n)!$ (c) $x_n = (2/n)^n$

   (d) $x_n = (2n+1)/(3n+4)^2$ (e) $x_n = e^n/n^n$ (f) $x_n = 2^{-n/2}$.

2. Let $|\phi\rangle = (x_1\,, x_2\,, x_3\,, \ldots) \in \ell^2$. Consider operators $A_1\,, A_2\,, A_3\,, A_4\,, A_5$ whose action on an element of the linear space is given by

$$A_1|\phi\rangle = (x_2\,, x_3\,, x_4\,, \ldots)\,,$$

$$A_2|\phi\rangle = (0\,, x_1\,, x_2\,, x_3\,, \ldots)\,,$$

$$A_3|\phi\rangle = (1!\,x_1\,, 2!\,x_2\,, 3!\,x_3\,, \ldots)\,,$$

$$A_4|\phi\rangle = (x_1/1!\,, x_2/2!\,, x_3/3!\,, \ldots)\,.$$

$$A_5|\phi\rangle = (2x_1\,, x_2\,, x_3\,, \ldots)\,.$$

   Indicate whether the operators $A_1 \ldots, A_5$ (i) have inverses; and (ii) are bounded; if so, find the value of the norm of the operator concerned.

3. Identify the functions that belong to $L^2(-\infty, \infty)$,:

   (a) $f(x) = (x^2+1)^{-1/4}$ (b) $f(x) = e^{-x}\,\cos\,x$ (c) $f(x) = e^{-1/x^2}$

   (d) $f(x) = (\sin\,x)/x$ (e) $f(x) = x^3\,e^{-x^2}$ (f) $f(x) = (\tanh\,x)/x$.

4. In $L^2(-\infty, \infty)$, find the adjoints of the following operators:

(a) $\dfrac{d}{dx}\, x$ (b) $\dfrac{d^2}{dx^2} + k^2$ ($k =$ real constant) (c) $x\, \dfrac{d}{dx}$

(d) $x\, \dfrac{d}{dx}\, x$ (e) $x^2\, \dfrac{d^2}{dx^2}$ (f) $\exp\left(ia\, \dfrac{d}{dx} \right)$ ($a =$ real constant) .

5. Consider the function $f(x) = x$ in the interval $[-\pi, \pi]$. Obtain the Fourier series for this function and hence show that $\sum_{n=0}^{\infty} (-1)^n/(2n+1) = \pi/4$.

6. Find the Fourier transform for the function $f(x) = A\, \exp(-|x|)$. Explicitly, verify that Parseval's theorem holds.

————-