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PH3520 Quantum Physics **Problem Set 3** **10.2.2010**

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1. The state of a three-dimensional particle is given by the following wavepacket in the momentum basis

$$\widetilde{\psi}(\mathbf{p}) \propto \exp\Big(-\frac{\alpha|\mathbf{p}|}{\hbar}\Big) .$$

   (a) Normalise the above wavefunction.

   (b) Obtain the wavefunction in the position basis.

   (c) Compute Δx, Δp_x and hence the product $\Delta x \Delta p_x$ for this state.

2. The conservation of probability in quantum mechanics can be cast as a local conservation law of the form

$$\frac{\partial \rho(\mathbf{x},t)}{\partial t} + \nabla \cdot \mathbf{S} = 0 ,$$

   where $\rho(\mathbf{x},t) \equiv |\psi(\mathbf{x},t)|^2$ is the probability density and $\mathbf{S}$ the probability current associated with the wavefunction $\psi(\mathbf{x},t)$.

   (a) Show that $\mathbf{S}$ for a particle whose time-evolution is given by the Hamiltonian $H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x})$ is given by

$$\mathbf{S} = \frac{\hbar}{2mi}\left(\psi^* \nabla \psi - \psi \nabla \psi^*\right) .$$

   (b) Show that, for a one-dimensional wavepacket, associated with particle of mass m

$$\int_{-\infty}^{\infty} S dx = \frac{\langle p \rangle}{m}$$

   Generalise to the case of a three-dimensional wavepacket.

   (c) Calculate $\mathbf{S}$ for the wave-function

$$\psi(\mathbf{x}) = \mathcal{N}\,\frac{e^{ikr}}{r} ,$$

   where $r^2 = x^2 + y^2 + z^2$ and $\mathcal{N}$ is a normalisation constant. Examine $\mathbf{S}$ for large values of $\mathbf{r}$ and interpret your result.

3. Consider a particle free to move in one dimension (with coordinate x). At time $t = 0$, the particle's state (a wavepacket) is specified by the normalised wavefunction (in the position basis)

$$\psi(x, 0) = \frac{1}{(\pi\sigma^2)^{1/4}} \; e^{ik_0 x} \; e^{-(x-x_0)^2/2\sigma^2}$$

   (a) Obtain the wavefunction in the momentum basis.

   (b) Obtain the wavefunction at a time $t > 0$?

   (c) Hence, evaluate $\langle \hat{x} \rangle$ as well as $\langle \hat{p} \rangle$ at time t and show that

$$\langle \hat{x} \rangle = x_0 + \frac{\langle \hat{p} \rangle t}{m} \; .$$

   (d) Evaluate Δx and Δp. Notice that while the state at time $t = 0$ has minimum uncertainty, that is no longer true at later times.

4. The time-dependent Schrödinger equation for a linear simple harmonic oscillator is, in the coordinate basis

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2} + \frac{1}{2} m\omega^2 x^2 \psi(x,t) \; .$$

Verify that the solution to this equation is given by

$$\psi(x,t) = \int_{-\infty}^{\infty} dx' \; K(x, x'; t) \; \psi(x', 0)$$

where

$$K(x, x'; t) = \left(\frac{m\omega}{2\pi i\hbar \sin \omega t} \right)^{1/2} \; \exp \left[\frac{im\omega}{2\hbar \sin \omega t} \{ (x^2 + x'^2) \cos \omega t - 2xx' \} \right] \; .$$

First check that $\psi(x,t)$ obeys the time-dependent Schrödinger equation. Then, check to see whether the proper initial condition is satisfied at $t = 0$, i.e., $\psi(x,t)$ reduces to the *given* initial function $\psi(x, 0)$. $K(x, x'; t)$ is called the *propagator* because it takes you from a solution at time $t = 0$ to a solution at subsequent instants of time t. Note that $\psi(x,t)$ depends on $\psi(x', 0)$ at *all* points x'.