DEPARTMENT OF PHYSICS
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PH3520 Quantum Physics **Problem Set 4** **25.2.2011**

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1. The radial momentum operator of a particle moving in three dimensions is defined in quantum mechanics as

$$p_r = \frac{1}{2}\left(\mathbf{p}\cdot\frac{\mathbf{r}}{r} + \frac{\mathbf{r}}{r}\cdot\mathbf{p}\right),$$

where $\mathbf{r}$ and $\mathbf{p}$ are its position and momentum operators, respectively. Clearly, p_r is a Hermitian operator.

   (a) Using the fact that $\mathbf{p} = -i\hbar\,\nabla$ in the position representation, show that p_r is represented by the differential operator

$$p_r = -i\hbar\left(\frac{\partial}{\partial r} + \frac{1}{r}\right).$$

   (b) Hence find the differential operator representing p_r^2.

2. (a) Express the Cartesian components L_x, L_y, L_z of the orbital angular momentum operator $\mathbf{L} = \mathbf{r}\times\mathbf{p}$ of a particle moving in space, in terms of the spherical polar coordinates $(r\,,\,\theta\,,\,\varphi)$ and the corresponding derivative operators $(\partial/\partial r\,,\,\partial/\partial\theta\,,\,\partial/\partial\varphi)$.

   (b) Hence find the representation for the operator $\mathbf{L}^2$ in spherical polar coordinates.

   (c) Show that the operator for the kinetic energy, T, can be written as

$$T = \frac{\mathbf{p}^2}{2m} = \frac{1}{2m}\left(p_r^2 + \frac{\mathbf{L}^2}{r^2}\right).$$

3. Obtain the probability density that an electron in the ground state of the hydrogen atom is found at a distance between r and $r+\Delta r$. (Note that you will have to take into account the nontrivial integration measure in spherical polar coordinates.) Show that this probability density has a maximum at the Bohr radius. What is the probability of finding the electron at a radius greater than the Bohr radius?

4. Show that the uncertainty relation for Δx and Δp_x is obeyed by an electron in the ground state of the hydrogen atom.

5. Repeat parts 3 and 4 for an electron in the $2s$ and $2p$ states of the hydrogen atom.

6. Consider an electron in the energy eigenstate $|n, l, m\rangle$ of the hydrogen atom. Show that

$$\langle T \rangle = -\frac{1}{2}\langle V \rangle \quad ,$$

where T and V are respectively the kinetic and potential energy of the electron. This is the quantum version of the *classical virial theorem*, which states that if the potential $V \sim r^k$, then the averages $\overline{T}$ and $\overline{V}$ are related by $\overline{T} = k\overline{V}/2$. Hence, show that ($a_0$ is the Bohr radius)

$$\left\langle \frac{1}{r} \right\rangle = \frac{1}{a_0 n^2} \quad .$$

7. The Hamiltonian of the hydrogen atom is, in suitable units,

$$H = \frac{\mathbf{p}^2}{2m} - \frac{e^2}{r} \quad .$$

The quantum mechanical version of the Runge-Lenz vector $\mathbf{A}$ is given by

$$\mathbf{A} = \frac{1}{2m}(\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) - e^2 \frac{\mathbf{r}}{r} \quad ,$$

where the ordering ambiguity in the first term has been taken care of by appropriate (anti)symmetrisation. Show that $[H, \mathbf{A}] = 0$.