

DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH3520 Quantum Physics

Problem Set 6

28.03.2011

The quantum ideal gas

1. Show that the grand partition function for a quantum ideal gas with chemical potential μ in a volume V at temperature T is given by

$$\log Q(V, T, \mu) = -\eta \frac{2\pi g V (2m)^{3/2}}{h^3} \int_0^\infty d\varepsilon \varepsilon^{1/2} \log(1 - \eta e^{\beta(\mu - \varepsilon)}) ,$$

where $\eta = +1$ for bosons and -1 for fermions; $g = (2s + 1)$ is the internal degeneracy due to the spin of the particle. Note that $\eta = \exp(2\pi i s)$ as a consequence of the spin-statistics theorem.

2. By computing the pressure, show that the equation of state for the quantum gas can be written as

$$\frac{PV}{k_B T} = \log Q(V, T, \mu) .$$

3. By integrating by parts, show that one can rewrite the grand partition function and hence the grand potential¹ can be written as

$$\Phi(V, T, \mu) = -\frac{4\pi g V (2m)^{3/2}}{3h^3} \int_0^\infty \frac{d\varepsilon \varepsilon^{3/2}}{e^{\beta(\varepsilon - \mu)} - \eta} .$$

(Verify that the surface term obtained on integrating by parts vanishes.)

4. Using the above result, show that

$$U \equiv \langle E \rangle = -\frac{\partial}{\partial \beta} \log Q = \frac{2\pi g V (2m)^{3/2}}{h^3} \int_0^\infty \frac{d\varepsilon \varepsilon^{3/2}}{e^{\beta(\varepsilon - \mu)} - \eta} ,$$
$$P = -\frac{\partial \Phi}{\partial V} = \frac{4\pi g (2m)^{3/2}}{3h^3} \int_0^\infty \frac{d\varepsilon \varepsilon^{3/2}}{e^{\beta(\varepsilon - \mu)} - \eta} .$$

It is easy to see that $P = \frac{2}{3} \frac{U}{V} = \frac{2}{3} u$, thus proving the desired relation.

¹The grand potential(or function) is defined as follows in statistical mechanics:
 $\Phi(V, T, \mu) = -k_B T \log Q(V, T, \mu)$.

5. Using $\langle N \rangle = -\frac{\partial \Phi}{\partial \mu}$, show that

$$\frac{3\sqrt{\pi}n\lambda^3}{4\beta^3} = \frac{3nh^3}{4\pi(2m)^{3/2}} = \int_0^\infty \frac{d\varepsilon \varepsilon^{3/2} \beta e^{\beta(\varepsilon-\mu)}}{(e^{\beta(\varepsilon-\mu)} - \eta)^2} .$$

6. Consider the function $g(\varepsilon) = \frac{\beta e^{\beta(\varepsilon-\mu)}}{(e^{\beta(\varepsilon-\mu)} + 1)^2}$ – this is the derivative of the Fermi function. Show that it is sharply peaked about $\varepsilon = \mu$ at low temperature and can be replaced by $\delta(\varepsilon - \mu)$ at zero temperature. By studying $\log g(\varepsilon)$, argue that one can carry out the **approximation**:

$$g(\varepsilon) \xrightarrow{T \rightarrow 0} \left[\frac{\beta}{2\sqrt{\pi}} e^{-\beta^2(\varepsilon-\mu)^2/4} \right] \xrightarrow{T=0} \delta(\varepsilon - \mu) .$$

Hence, show that for a Fermi gas, one has

$$\varepsilon_F^{3/2} \equiv \frac{3nh^3}{4\pi(2m)^{3/2}} \sim \int_{-\infty}^\infty d\varepsilon \varepsilon^{3/2} \frac{\beta}{2\sqrt{\pi}} e^{-\beta^2(\varepsilon-\mu)^2/4} .$$

7. By expanding $\varepsilon^{3/2}$ about μ to second order, one can carry out the Gaussian integrations to obtain

$$\varepsilon_F^{3/2} = \mu^{3/2} + \frac{3k_B^2 T^2}{4\mu^{1/2}} + \mathcal{O}(T^3) .$$

Hence, one sees that the chemical potential reduces as temperature increases.

$$\mu(n, T) = \varepsilon_F - \gamma (k_B^2 T^2 / \varepsilon_F) + \mathcal{O}(T^3) ,$$

where γ is a positive constant. Determine γ . (Compare with the exact answer: $\gamma = \pi^2/8$)