

DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH5460 Classical Field Theory Assignment 3 9.8.2012 (due: 16.8.2012)

(Mostly) Finite Groups

1. Explicitly write out the multiplication tables for the groups, $\mathbb{Z}_2 \times \mathbb{Z}_2$ and \mathbb{Z}_4 , both of order 4. Hence, show that they are **not** isomorphic to each other.
2. The dihedral group, D_{2n} , is generated by two elements x and y satisfying the relations

$$x^n = 1, \quad y^2 = 1, \quad \text{and} \quad yx = x^{n-1}y. \quad (*)$$

- (a) By working out the multiplication table for D_3 , prove the isomorphism, $D_6 \sim S_3$.
 - (b) Obtain subgroups, with order 2 and 3, of S_3 and check to see if they are *normal* subgroups. If they are normal, obtain the group multiplication law for the corresponding coset. Is S_3 a simple group?
3. Consider the cyclic group $\mathbb{Z}_3 = (e, g, g^2)$ with $g^3 = e$. The regular representation is a map from the group to 3×3 matrices with g being represented by the matrix

$$g \longrightarrow M(g) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

with the property that $M(g_1)M(g_2) = M(g_1 \cdot g_2)$ for any $g_1, g_2 \in \mathbb{Z}_3$.

- (a) What does the identity $g^3 = e$ become in this representation? Verify that it holds.
 - (b) Obtain the similarity transformation that diagonalises $M(g)$.
4. Obtain the condition under which the 2×2 matrix U given below is an $SU(2)$ matrix.

$$U = \alpha_0 I + i\alpha_i \sigma^i, \quad ,$$

where σ^i are the Pauli sigma matrices and α_0, α_i are real parameters. What is the geometric interpretation of this condition?

5. Since the Pauli sigma matrices form a basis for traceless 2×2 hermitian matrices, an arbitrary $SU(2)$ matrix (in the two dimensional representation) can also be written as

$$U^{(2)}(\hat{n}, \theta) = \exp\left(i\frac{\theta}{2}\hat{n} \cdot \vec{\sigma}\right) \quad ,$$

where $|\hat{n}| = 1$. Solve for α_0, α_i in terms of θ and \hat{n} . Hence show that $U(\hat{n}, 2\pi) = -I$.

6. **This question will not be graded.** List out all finite groups of order ≤ 10 and identify all the simple groups in that list.