DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH5460 Classical Field Theory Assignment 6 17.9.2012 (due: 24.9.2012)

Symmetries and Conserved Currents: The Vector Field

Under general coordinate transformations $x'^{\mu} = x'^{\mu}(x)$, a vector field transforms as

$$A'_{\mu}(x') = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \, A_{\nu}(x)$$

The Lagrangian density describing the dynamics of a (massive) vector field, called the Proca Lagrangian density, is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_{\mu} A^{\mu}$$

The action corresponding to this Lagrangian density is invariant under the Poincaré group: i.e., the set of constant translations:

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu}$$

and Lorentz transformations

$$x'^{\mu} = x^{\mu} + \lambda^{\mu}{}_{\nu} \, x^{\nu}$$

where ϵ^{μ} and $\lambda^{\mu\nu} = -\lambda^{\nu\mu} \equiv \lambda^{\mu}{}_{\tau} \eta^{\tau\nu}$ are infinitesimal constant parameters. In addition, when $m = 0$, the action is invariant under the *local gauge transformation*

$$\delta A_{\mu} = \partial_{\mu} \alpha$$

with $\delta x^{\mu} = 0$ and α is a function of x.

1. The Noether current associated with translations is the energy-momentum (or stress) tensor $T^{\mu\nu}$. Show that the Noether current is

$$T^{\rho\mu} = \frac{\delta \mathcal{L}}{\delta \partial_{\rho} A_{\sigma}} \partial^{\mu} A_{\sigma} - \eta^{\rho\mu} \mathcal{L}$$

   Verify that it is not symmetric but is conserved i.e., $\partial_{\rho} T^{\rho\mu} = 0$.

2. Show that the Noether current associated with the Lorentz transformation $\lambda_{\mu\nu}$ is

$$\widetilde{M}^{\rho\mu\nu} = L^{\rho\mu\nu} + \left(\frac{\delta \mathcal{L}}{\delta \partial_{\rho} A_{\mu}} A^{\nu} - \frac{\delta \mathcal{L}}{\delta \partial_{\rho} A_{\nu}} A^{\mu} \right) \, ,$$

where $L^{\rho\mu\nu} \equiv (x^\mu T^{\rho\nu} - x^\nu T^{\rho\mu})$ is the (orbital) angular momentum that we obtained for a scalar field[1]. Note that the Noether current for the vector field $\widetilde{M}^{\rho\mu\nu} \neq L^{\rho\mu\nu}$. Further, $\partial_\rho L^{\rho\mu\nu} = T^{\mu\nu} - T^{\nu\mu} \neq 0$ since $T^{\mu\nu}$ is not symmetric. However, verify that $\partial_\rho \widetilde{M}^{\rho\mu\nu} = 0$ as is expected for a Noether current.

Obtaining a symmetric energy-momentum tensor

We will construct a symmetric energy-momentum tensor $\Theta^{\mu\nu}$ with the following properties:

(i) It is symmetric and is conserved;

(ii) the total energy and momentum as computed from $\Theta^{\mu\nu}$ is the same as those computed from $T^{\mu\nu}$;

(iii) the angular momentum density defined by $M^{\rho\mu\nu} \equiv (x^\mu \Theta^{\rho\nu} - x^\nu \Theta^{\rho\mu})$ is conserved;

(iv) the conserved charges from $M^{\rho\mu\nu}$ agree with those obtained from $\widetilde{M}^{\rho\mu\nu}$.

Property (ii) in the above list can be accomplished if $\Theta^{\mu\nu} = T^{\mu\nu} + \partial_\rho H^{\rho\mu\nu}$ and the surface terms in the charges vanish. Further, suppose $H^{\rho\nu\mu} = -H^{\nu\rho\mu}$ i.e., it is antisymmetric in the first two indices. Then, the conservation part of (ii): $\partial_\mu \Theta^{\mu\nu} = \partial_\mu T^{\mu\nu} = 0$ follows from the antisymmetry of $H^{\rho\nu\mu}$ which implies that $\partial_\rho \partial_\nu H^{\rho\nu\mu} = 0$. Property (iv) will be true if $M^{\mu\nu\rho}$ and $\widetilde{M}^{\mu\nu\rho}$ differ by a total derivative whose surface terms in the charges vanish.

3. Show that $H^{\rho\mu\nu} = -F^{\rho\mu} A^\nu$ leads to a symmetric energy-momentum tensor with properties (i)-(iii).

4. Show that the energy density Θ^{00} is positive definite. Further, show that Θ^{i0} is the Poynting vector.

5. Find the trace of the energy-momentum tensor i.e., $\mathrm{Tr}(\Theta) \equiv \Theta^{\mu\nu}\eta_{\mu\nu}$. What happens to it when $m = 0$?

6. Show that property (iv) above is true.

The angular momentum $M^{\mu\nu\rho}$ consists of two terms: the orbital part $L^{\mu\nu\rho}$ (as defined earlier) and the spin-part defined by $M^{\mu\nu\rho} = L^{\mu\nu\rho} + S^{\mu\nu\rho}$.

7. Show that $S^{0ij} = A^i \pi^j - A^j \pi^i$ where $\pi^i = \frac{\delta \mathcal{L}}{\partial_0 A_i}$ is the momentum canonical conjugate to A_i with $i, j = 1, 2, 3$.

[1] We call this the orbital angular momentum in analogy with the definition of the orbital angular momentum $\mathbf{L} = \mathbf{x} \times \mathbf{p}$ for a particle.