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PH5480 Quantum Field Theory — Problem Set 0 — Jan 2013

1. A Lorentz transformation of spacetime

$$\widetilde{x}^\mu = \Lambda^\mu{}_\nu \, x^\nu \ ,$$

is an orthogonal transformation satisfying

$$\Lambda^{\mathrm{T}}{}_\rho{}^\mu \ \eta_{\mu\nu} \ \Lambda^\nu{}_\sigma = \eta_{\rho\sigma} \ , \tag{1}$$

where η is the diagonal matrix, $\mathrm{Diag}(1, -1, -1, -1)$. Show that

   (a) $\det(\Lambda)^2 = 1$.

   (b) Λ^{-1} is also a Lorentz transformation.

   (c) Let Λ_1 and Λ_2 be two Lorentz transformations. Then $\Lambda_1 \cdot \Lambda_2$ and $\Lambda_2 \cdot \Lambda_1$ are also Lorentz transformations. Also verify that the product of two orthochronous transformations is another orthochronous transformation.

   (d) Starting from the transformation rule for a fourth-rank tensor, show that the Levi-Civita tensor transforms as follows:

$$\widetilde{\epsilon}^{\mu\nu\rho\sigma} = \det(\Lambda) \ \epsilon^{\mu\nu\rho\sigma} \ .$$

   Thus it is invariant under proper Lorentz tranformations but changes sign under parity and time-reversal.

2. **Infinitesimal Lorentz Transformations:** Consider proper orthochronous Lorentz transformations – they are continuous transformations and one can consider infinitesimal version of such tranformations. Let

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \lambda^\mu{}_\nu + O(\lambda^2) \ .$$

   (a) Show that the orthogonality condition Eq. (1) implies

$$\lambda_{\mu\nu} \equiv \eta_{\mu\rho} \lambda^\rho{}_\nu = -\lambda_{\nu\mu} \ .$$

   (b) Identify the (infinitesimal) parameter for boosts in the x, y and z directions.

   (c) Identify the (infinitesimal) parameter for rotations about the x, y and z axes.

3. (a) Show that the Euler-Lagrange equation for the following Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \phi(x)\partial^\mu \phi(x) - \tfrac{m^2}{2}\phi(x)^2 \ ,$$

is the Klein-Gordon equation.

(b) Show that the Euler-Lagrange equation for the following Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) + 4\pi J_\mu(x)A^\mu(x) \ ,$$

where $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$ leads to the Maxwell equations when combined with the Bianchi identity.

4. **Second Quantization:** Let $a(\mathbf{x})$ and $a^\dagger(\mathbf{x})$ satisfy the following algebra:

$$[a(\mathbf{x}), a^\dagger(\mathbf{x}')] = \delta^3(\mathbf{x}-\mathbf{x}') \quad , \quad [a(\mathbf{x}), a(\mathbf{x}')] = [a^\dagger(\mathbf{x}), a^\dagger(\mathbf{x}')] = 0 \ .$$

Let $|0\rangle$ represent the vacuum state i.e., $a(\mathbf{x})|0\rangle = 0$ and further define:

$$|\psi\rangle \equiv \int d^3x_1 \cdots d^3x_N \ \psi(\mathbf{x}_1, \ldots, \mathbf{x}_N, t) a^\dagger(\mathbf{x}_1) \cdots a^\dagger(\mathbf{x}_N) \ |0\rangle \ .$$

Consider the operator

$$\hat{H} \equiv \int d^3x \ a^\dagger(\mathbf{x}) \Big[-\frac{\hbar^2}{2m}\nabla^2 + V_0(\mathbf{x}) \Big] a(\mathbf{x}) + \frac{1}{2}\int d^3x d^3y \ V_1(\mathbf{x}-\mathbf{y}) a^\dagger(\mathbf{x}) a^\dagger(\mathbf{y}) a(\mathbf{x}) a(\mathbf{y}) \ .$$

Show that $i\hbar\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi, t\rangle$ implies the following N-body time-dependent Schrödinger equation

$$i\hbar\frac{\partial \psi(\mathbf{x}_1, \ldots, \mathbf{x}_N, t)}{\partial t} = \Big[\sum_{i=1}^{N} \Big(-\frac{\hbar^2}{2m}\nabla_i^2 + V_0(\mathbf{x}_i) \Big) + \sum_{i=1}^{N}\sum_{j=1}^{i-1} V_1(\mathbf{x}_i - \mathbf{x}_j) \Big] \psi(\mathbf{x}_1, \ldots, \mathbf{x}_N, t)$$

Remark: This is largely meant as a review for students especially those who have not done Classical Field Theory.