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Quantum Field Theory — Problem Set 2 — Jan 2013

Consider the Lagrangian density for a real scalar field in $3+1$ dimensions

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \ .$$

One can expand $\phi(x)$ in the complete basis provided by all solutions to the equation of motion as

$$\phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \left(a(\mathbf{k})\ e^{-ik\cdot x} + a^\dagger(\mathbf{k})\ e^{ik\cdot x}\right)$$

with $k = (\omega_k, \mathbf{k})$ and $\omega_k = \sqrt{m^2 + \mathbf{k}\cdot\mathbf{k}}$. In quantizing the field theory, we treat $\phi(x)$ as an operator acting on a Hilbert space. Equivalently, we treat $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$ as operators. Assume the following (quantum) commutation relations (in natural units) among the $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$.

$$\left[a(\mathbf{k}), a^\dagger(\mathbf{k}')\right] = \delta^3(\mathbf{k}-\mathbf{k}')\ , \quad \left[a(\mathbf{k}), a(\mathbf{k}')\right] = 0\ , \quad \left[a^\dagger(\mathbf{k}), a^\dagger(\mathbf{k}')\right] = 0 \qquad (1)$$

1. Hence, show that the equal time commutators are given by

   $$[\phi(\mathbf{x},0), \pi(\mathbf{x}',0)] = i\delta^3(\mathbf{x}-\mathbf{x}')\ ,\ [\phi(\mathbf{x},0), \phi(\mathbf{x}',0)] = 0\ ,\ [\pi(\mathbf{x},0), \pi(\mathbf{x}',0)] = 0$$

   In the above, $\pi(x)$ is generalised momentum canonically conjugate to $\phi(x)$.

2. Verify that

   $$\begin{aligned} P_\mu &\equiv \int d^3x\ T_{0\mu} = \int d^3k\ k_\mu\ \left[a(\mathbf{k})a^\dagger(\mathbf{k}) + a^\dagger(\mathbf{k})a(\mathbf{k})\right] \\ M_{\mu\nu} &\equiv \int d^3x\ (x_\mu T_{0\nu} - x_\nu T_{0\mu}) = \int d^3k\ (x_\mu k_\nu - x_\nu k_\mu)\left[a(\mathbf{k})a^\dagger(\mathbf{k}) + a^\dagger(\mathbf{k})a(\mathbf{k})\right] \end{aligned}$$

3. Show that the commutators for the P_μ and $M_{\mu\nu}$ are indeed the realizations of the Lie algebra of the Poincaré group.

4. Now suppose that in Eq. (1), we replace commutators with anti-commutators. Of course, this implies that (ϕ, π) satisfies equal time anticommutators – check this. Then, show that the P_μ and $M_{\mu\nu}$ continue to satisfy the Lie algebra of the Poincaré group.