DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH5460 Classical Field Theory   Assignment 1   1.8.2014 (due: 8.8.2014)

---

1. A particle of mass m and charge q moves under the influence of an electromagnetic field. The electric and magnetic fields (not necessarily time-independent) can be written in terms of the scalar potential $\Phi(\mathbf{x})$ and vector potential $\mathbf{A}(\mathbf{x})$ as follows:

$$\mathbf{E} = -\nabla\Phi - \partial\mathbf{A}/\partial t \ , \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A} \ .$$

Show that the Euler-Lagrange equations for the Lagrangian given below reproduces the Lorentz force law:

$$L = \frac{1}{2} m \ \dot{\mathbf{x}}^2 + q \ \dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}) - q \ \Phi(\mathbf{x}) \ .$$

This is an example of a Lagrangian that is **not** of the form $T - V$ as it involves a velocity-dependent conservative force.

2. According to wikipedia, a **brachistochrone curve** (Greek: brachistos = the shortest, chronos = time), or curve of fastest descent, is the path that will carry a point-like body from one place to another in the least amount of time. The body is released at rest from the starting point and is constrained to move without friction along the curve to the end point, while under the action of constant gravity. The *calculus of variations* was first developed to solve this problem. The functional that has to be extermised is the *Time* functional, $T(\mathbf{x}(s))$, where $\mathbf{x}(s)$ for some $s \in [0, 1]$ parametrises the curve such that $\mathbf{x}(0)$ is the initial point and $\mathbf{x}(1)$ is the end point. With no loss of generality, assume that the motion is in the xz-plane with gravity acting in the z direction. Construct the functional, extremise it and solve it to obtain the brachistochrone curve.
`http://www-history.mcs.st-andrews.ac.uk/HistTopics/Brachistochrone.html` has some interesting historical details.

3. A free particle of unit mass moving in one dimension with coordinate q. We are given that the particle was $q = 1$ at $t = 0$ and at $q = 5$ at time $t = 1$. The action for this particle is

$$S[q(t)] = \frac{1}{2}\int_0^1 dt\,(\dot{q})^2 \ .$$

Consider the following four possible trajectories of the particles:

$$\begin{aligned} q_A(t) &= 2\sin(2\pi t) + 4t + 1 \ , \\ q_B(t) &= -3t^2 + 7t + 1 \ , \\ q_C(t) &= 4t + 1 \ , \\ q_D(t) &= -2\cos(\pi t) + 3 \ . \end{aligned}$$

Of the four paths, $q_C(t)$ is the solution to the equations of motion – this is usually called the *classical solution*. Compute the action for all four trajectories and verify that the minimum occurs for $q_C(t)$.

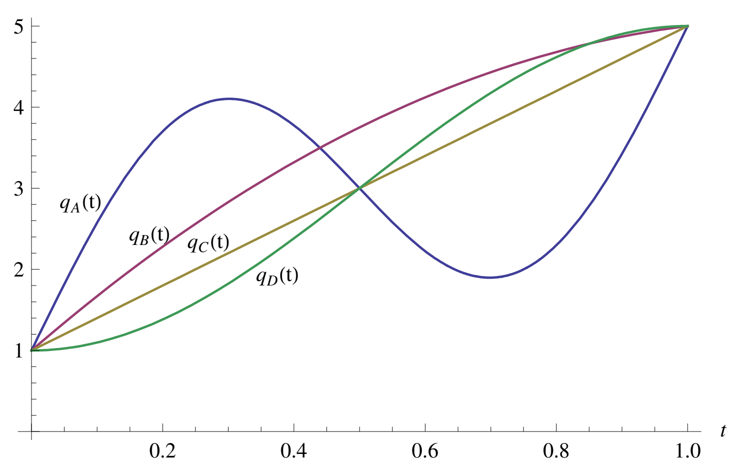

Figure 1: A plot of the four trajectories. Note that all paths begin at $q = 1$ and end at $q = 5$.

4. Consider the action for a particle of unit mass in a harmonic potential

$$S[q(t)] = \frac{1}{2}\int_0^1 dt\left[(\dot{q})^2 - \pi^2(q-3)^2\right] \ .$$

Evaluate the action for the four trajectories that were considered in the previous problem. Show that the minimum value occurs for the solution $q_D(t)$ and verify that it is indeed the classical solution.