DEPARTMENT OF PHYSICS

INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH5460 Classical Field Theory | Assignment 10 | 24.10.2014
Due: 31.10.2014

Consider the Lagrangian density in $3+1$ dimensions which is invariant under local $SO(3)$ transformations (λ and μ are positive and real.

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu a}F^a_{\mu\nu} + \frac{1}{2}\pi^{\mu a}\pi^a_\mu - \frac{1}{4}\lambda\left(\phi^a\phi^a - \frac{\mu^2}{\lambda}\right)^2 \quad ,$$

where $F^a_{\mu\nu} = (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e\epsilon^{abc}A^b_\mu A^c_\nu)$; and $\pi^a_\mu = \partial_\mu\phi^a + e\epsilon^{abc}A^b_\mu\phi^c$. $(a,b,c = 1,2,3)$

1. Obtain the Euler-Lagrange equations of motion. Also convince yourself that a ground state solution in the absence of gauge fields 'breaks' $SO(3)$ to $SO(2)$.

2. Consider the following time-independent ansatz

$$\begin{aligned} A^a_0 &= 0 \quad , \\ A^a_i &= \epsilon_{aij}x_j[1-K(r)]/er^2 \quad , \\ \phi^a &= x_a H(r)/er^2 \quad , \end{aligned}$$

   where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Substitute the ansatz in the equations of motion and show that the equations of motion reduce to

$$\begin{aligned} r^2 K'' &= K(K^2-1) + KH^2 \quad , \\ r^2 H'' &= 2HK^2 + \frac{\lambda}{e^2}(H^3 - C^2 r^2 H) \quad , \end{aligned}$$

   where $C = \mu e/\sqrt{\lambda}$ and the prime denotes differentiation w.r.t. r. Obtain the condition(s) on H and K such that the Hamiltonian (as well as the Lagrangian) is finite.

3. In the limit where $\lambda, \mu \to 0$ keeping the ratio $C = \mu e/\sqrt{\lambda}$ constant, show that $K = Cr/\sinh(Cr)$ and $H = Cr\coth(Cr) - 1$ solve the equations of motion. This limit is usually referred to as the *Prasad-Sommerfield* limit. We will call this solution as the Prasad-Sommerfield (PS) solution.

4. For large r, the PS solution reduces to the classical vacuum where the $SO(3)$ gauge symmetry is broken to $U(1)$. The field strength of this $U(1)$ field is given by

$$\mathcal{F}_{\mu\nu} = \partial_\mu(\hat{\phi}^a A^a_\nu) - \partial_\nu(\hat{\phi}^a A^a_\mu) - (1/e)\epsilon^{abc}\hat{\phi}^a\partial_\mu\hat{\phi}^b\partial_\nu\hat{\phi}^c \quad ,$$

   where $\hat{\phi}^a \equiv \phi^a/(\phi^b\phi^b)^{1/2}$. This definition has been chosen based on the observation that if after a gauge transformation $\phi^a = \delta^{a3}$ everywhere within a region, then $\mathcal{F}_{\mu\nu} = \partial_\mu A^3_\nu - \partial_\nu A^3_\mu$ in that region. Verify this observation. What is that gauge transformation for large r for the PS solution? Obtain the electric and magnetic charge of the soliton from this field strength for the PS solution? Compare the solution with that of the Dirac monopole in the large r region.

5. Obtain the mass of the soliton by first deriving the Hamiltonian density and then integrating it over all space in the usual way.