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PH5460 Classical Field Theory Assignment 4 27.8.2014

The most general $SU(2)$ matrix can be written as

$$S(\hat{n},\theta) = \exp(i\theta\tfrac{\hat{n}\cdot\vec{\sigma}}{2}) = \cos\tfrac{\theta}{2}\ \mathbf{1} + i\sin\tfrac{\theta}{2}\ \hat{n}\cdot\vec{\sigma}\ .$$

Let $D^{(1/2)}$ denote $S(\hat{e}_y,\theta)$ – this is the spin analogue of a rotation by an angle θ about the y-axis.

Recall the definition of the tensor (or Kronecker) product of two matrices, an $m\times m$ matrix $X=(x_{ij})$ and and a $n\times n$ matrix $Y=(y_{ab})$. The direct product $X\otimes Y$ is the following $mn\times mn$ matrix written as $n\times n$ blocks.

$$X\otimes Y = \begin{pmatrix} x_{11}Y & x_{12}Y & \cdots & x_{1m}Y \\ x_{21}Y & x_{22}Y & \cdots & x_{2m}Y \\ \vdots & \vdots & \cdots & \vdots \\ x_{m1}Y & x_{m2}Y & \cdots & x_{mm}Y \end{pmatrix} .$$

Note that generically, $X\otimes Y \neq Y\otimes X$.

Let us represent the Clebsch-Gordan coefficients for all possible values of j by the matrix $U^{(j_1,j_2)}$ with first $(2j_1+2j_2+1)$ rows containing the CG coefficients for $j=(j_1+j_2)$ further ordered by decreasing m, the next $(2j_1+2j_2-1)$ for the CG coefficients for the next lower value of j i.e., (j_1+j_2-1) and so on.

$$U^{(j_1,j_2)} := \langle j,m'|j_1,m_1';j_2,m_2'\rangle$$

For instance, when $j_1=j_2=1/2$, the first three rows correspond to the $j=1$ ($m=1,0,-1$) and the fourth row for $j=m=0$. One can show that

$$U^{1/2,1/2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \quad , \quad U^{1,1/2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} & 0 \end{pmatrix} .$$

(The above paragraph is given so that you can connect with your QM Course)

Using your favourite symbolic manipulation package (mathematica/maxima/...) verify:

1. Compute $D^{(1/2)}\otimes D^{(1/2)}$. Then, verify that $U^{1/2,1/2}\cdot(D^{1/2}\otimes D^{1/2})\cdot(U^{1/2,1/2})^\dagger$ is block diagonal with a 3×3 block and another 1×1 block. Call the 3×3 block, $D^{(1)}$. Does it look like a rotation about the y-axis – if not, find the similarity transformation that does that.

2. Compute $D^{(1)}\otimes D^{1/2}$. Then, verify that $U^{1,1/2}\cdot(D^{(1)}\otimes D^{(1/2)})\cdot(U^{1,1/2})^\dagger$ is block diagonal with a 4×4 block and another 2×2 block.

3. Repeat the above computations by replacing $D^{(1/2)}$ by S. Does the block diagonal nature that you saw persist?