**DEPARTMENT OF PHYSICS**
**INDIAN INSTITUTE OF TECHNOLOGY, MADRAS**

PH5460 Classical Field Theory    Assignment 5    27.8.2014 (due: 3.9.2014)

## (Mostly) Finite Groups

1. Explicitly write out the multiplication tables for the groups, $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_4$, both of order 4. Hence, show that they are **not** isomorphic to each other.

2. The dihedral group, $D_{2n}$, is generated by two elements $x$ and $y$ satisfying the relations

$$x^n = 1, \quad y^2 = 1, \quad \text{and} \quad yx = x^{n-1} y \ . \qquad (*)$$

   (a) By working out the multiplication table for $D_3$, prove the isomorphism, $D_6 \sim S_3$.

   (b) Obtain subgroups, with order 2 and 3, of $S_3$ and check to see if they are *normal* subgroups. If they are normal, obtain the group multiplication law for the corresponding coset. Is $S_3$ a simple group?

3. Consider the cyclic group $\mathbb{Z}_3 = (e, g, g^2)$ with $g^3 = e$. The regular representation is a map from the group to $3 \times 3$ matrices with $g$ being represented by the matrix

$$g \longrightarrow M(g) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \ ,$$

   with the property that $M(g_1)M(g_2) = M(g_1 \cdot g_2)$ for any $g_1, g_2 \in \mathbb{Z}_3$.

   (a) What does the identity $g^3 = e$ become in this representation? Verify that it holds.

   (b) Obtain the similarity transformation that diagonalises $M(g)$.

4. Obtain the condition under which the $2 \times 2$ matrix $U$ given below is an $SU(2)$ matrix.

$$U = \alpha_0 I + i\alpha_i \sigma^i \quad ,$$

   where $\sigma^i$ are the Pauli sigma matrices and $\alpha_0, \alpha_i$ are real parameters. What is the geometric interpretation of this condition?

5. Since the Pauli sigma matrices form a basis for traceless $2 \times 2$ hermitian matrices, an arbitrary $SU(2)$ matrix (in the two dimensional representation) can also be written as

$$U^{(2)}(\hat{n}, \theta) = \exp\left(i\frac{\theta}{2}\hat{n} \cdot \vec{\sigma}\right) \quad ,$$

where $|\hat{n}| = 1$. Solve for $\alpha_0, \alpha_i$ in terms of $\theta$ and $\hat{n}$. Hence show that $U(\hat{n}, 2\pi) = -I$.

6. **This question will not be graded.** List out all finite groups of order $\leq 10$ and identify all the simple groups in that list.