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PH5460 Classical Field Theory   Assignment 9   15.10.2014 (due: 23.10.2014)

Non–abelian gauge theories

Let $\Phi = (\phi_1, \phi_2, \cdots, \phi_r)^T$ be a field in a representation R (dim $R = r$) of a Lie group G (dim $G = N$). The field Φ may have other indices which are suppressed for ease of presentation. In order to have a concrete example in mind, choose $\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ to be in the fundamental representation of the Lie group $SU(2)$ (i.e., the $j = \frac{1}{2}$ of $SU(2)$). In this example, $N = 3$ and $r = 2$.

Let T_a, $a = 1, \cdots, N$ be the generators of the Lie algebra in the representation R. By definition, they are $r \times r$ *Hermitian matrices* which satisfy the Lie algebra G

$$[T_a, T_b] = i f_{ab}{}^c \, T_c \quad , \tag{1}$$

where $f_{ab}{}^c$ are the structure constants of the Lie algebra. The index structure of the matrix is $T_i^{\ j}$, $i, j = 1, \cdots, r$. Note that we use the indices from the beginning of the alphabet, $a, b, c, \cdots$ for the generator index and and the middle letters $i, j, k, \cdots$ for the representation index.

Exercise 1: Given that the generators of the Lie algebra satisfy the Jacobi identity, obtain a condition on the structure constants of the Lie algebra. Explicitly verify that this is true for the case of $SU(2)$.

Exercise 2: In the adjoint representation, $r = N$. Thus the representation indices and the generator indices are the 'same'. In this representation, the T_a have a simple presentation in terms of the structure constants:

$$(T_a)_b{}^c = -i f_{ab}{}^c \quad .$$

Show that the above T_a satisfy the Lie algebra G.

In the example we are considering, in the fundamental representation of $SU(2)$, the generators $T_a = \frac{1}{2}\sigma_a$, where σ_a are the Pauli matrices and the structure constants $f_{ab}{}^c = \epsilon_{ab}{}^c$. The a, b type indices are raised and lowered using the *metric*

$$h_{ab} \equiv 2 \ \mathrm{Tr}_F(T_a T_b)$$

where the the subscript F refers to choosing the matrices T_a in the fundamental representation of the Lie algebra. One can show that in our example $h_{ab} =$

δ_{ab}. One can choose the generators T_a such that the structure constant f_{abc} is completely *antisymmetric* in its three indices.

Under a local gauge transformation given by a group element g, the field Φ transforms as $\Phi' = g\ \Phi$ which in components is written as

$$\phi_i'(x) = g(x)_i^{\ j}\ \phi_j(x) \quad . \tag{2}$$

For a Lie group, the group element g can be written as

$$g(x) = \exp[i\ \theta^a(x)\ T_a] \quad , \tag{3}$$

where $\theta^a(x)$ are N parameters (angles) which specify a group transformation. For infinitesimal transformations parametrised by $\delta\theta^a$, one obtains

$$g(x) = \mathbf{1} + i\delta\theta^a(x)\ T_a + \mathcal{O}(\delta\theta^2) \quad . \tag{4}$$

Note that by $\mathbf{1}$ in the above equation, we mean the $r \times r$ identity matrix.

It is of interest to write Lagrangians which are locally gauge invariant. The simplest method is to consider a Lagrangian which is globally G-invariant i.e., $\mathcal{L}(\Phi', \partial_\mu\Phi') = \mathcal{L}(\Phi, \partial_\mu\Phi)$, where $\Phi' = g\ \Phi$. This Lagrangian is typically not locally gauge invariant. This has to do with the fact that $\partial_\mu\Phi$ does not transform nicely (covariantly) under local gauge transformations. Explictly, one obtains

$$\partial_\mu\Phi' = g\ (\partial_\mu\Phi) + (\partial_\mu g)\ \Phi \quad . \tag{5}$$

A *minimal prescription* is to replace all derivatives of Φ with *covariant derivatives* of Φ. These are defined by

$$D_\mu\Phi \equiv \Big(\partial_\mu - i\ [A_\mu^a T_a]\Big)\ \Phi \quad , \tag{6}$$

where we have introduced a new field called the *gauge field* or the *connection*. In the above equation the term in the square brackets can be written in terms of a matrix valued gauge field $\mathcal{A}_\mu \equiv [A_\mu^a T_a]$.

Exercise 3: Show that for the case of $SU(2)$, where $T_a = \frac{1}{2}\sigma_a$, the matrix valued gauge field takes the form

$$\mathcal{A}_\mu = \frac{1}{2}\begin{pmatrix} A_\mu^3 & A_\mu^1 - iA_\mu^2 \\ A_\mu^1 + iA_\mu^2 & -A_\mu^3 \end{pmatrix}$$

The transformation of the covariant derivative of Φ can be chosen to be **identical** to that of a simple derivative under global transformations:

$$D_\mu'\phi'(x) = g(x)\ D_\mu\phi(x) \quad . \tag{7}$$

This fixes the tranformation of the gauge field to be

$$\mathcal{A}'_\mu = g\mathcal{A}_\mu g^{-1} + i\ g\partial_\mu g^{-1} \quad . \tag{8}$$

Note that here g need not be infinitesimal. Thus the Lagrangian $\mathcal{L}(\Phi, D_\mu\Phi)$ is clearly invariant under local gauge transformations.

Exercise 4: Derive the transformation of the gauge field A^a_μ (given in the above equation) under local gauge transformations. What is the tranformation of the gauge field under infinitesimal gauge transformations? This result can be identified with the covariant derivative of $\delta\theta^a$ in some representation. Identify the representation. When $G = U(1)$, compare the infinitesimal as well as the exact versions. Is this always true? Explain.

Exercise 5: Consider the example where Φ is in the two dimensional representation of $SU(2)$. Write the most general Lagrangian which is invariant under global $SU(2)$ transformations and involves upto two derivatives. Obtain the Noether current corresponding to this transformation.

Exercise 6: Now make the Lagrangian locally gauge invariant by replacing all derivatives by covariant derivatives. Regroup the locally gauge invariant Lagrangian into terms independent of the gauge field, terms linear in the gauge field and terms quadratic in the gauge field, i.e.,

$$\mathcal{L}(\Phi, D_\mu\Phi) = \mathcal{L}(\Phi, \partial_\mu\Phi) + \mathrm{Tr}_R(L_1^\mu \mathcal{A}_\mu) + \mathrm{Tr}_R(L_2^{\mu\nu}\mathcal{A}_\mu\mathcal{A}_\nu) \quad .$$

Is there any relation between L_1^μ and the Noether current derived in the previous exercise?

The locally gauge invariant Lagrangian which we have constructed so far does not involve any derivatives of the gauge field. We would like to include new terms which involve derivatives of the gauge field subject to the conditions that the terms are Lorentz scalars and are gauge invariant. For the case of electromagnetism the only gauge invariant object is the field strength which was used to construct the new terms to be added to the Lagrangian. A simple way to obtain the analog of the field strength for a generic group G is to use the following definition

$$\mathcal{F}_{\mu\nu}\ \Phi \equiv i[D_\mu, D_\nu]\ \Phi \tag{9}$$

Exercise 7: Show that

$$\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - i[\mathcal{A}_\mu, \mathcal{A}_\nu] \tag{10}$$

Exercise 8: Show that under gauge transformations, the field strength $\mathcal{F}_{\mu\nu}$ is generically not invariant but *covariant*, with the following tranformation law:

$$\mathcal{F}'_{\mu\nu} = g \; \mathcal{F}_{\mu\nu} \; g^{-1} \quad . \tag{11}$$

This implies that $\mathcal{F}$ transforms in the adjoint representation of the group G. Show that for $G = U(1)$, this implies that the field strength is invariant. Further, derive the infinitesimal transformation law from the finite one.

Exercise 9: Show that following are only two possible Lorentz and gauge invariant terms which can be constructed from this field strength keeping in mind the restriction on the number of derivatives.

(i) $\mathrm{Tr}_F(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu})$.

(ii) $\epsilon^{\mu\nu\rho\sigma} \; \mathrm{Tr}_F(\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma})$.

It can be shown that term (ii) is a total derivative and can be written as $\partial_\mu \Omega^\mu$. Obtain an expression for Ω^μ. Ω_μ is called the *Chern–Simons* term. This plays a special role in three dimensional gauge theories and in the context of the quantum Hall effect.

Now one can write a gauge invariant Lagrangian which includes derivatives of the gauge field as:

$$\begin{aligned} \mathcal{L}_{total} &= -\frac{1}{4e^2} \, \mathrm{Tr}_F(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}) + \mathcal{L}(\Phi, D_\mu\Phi) \quad , \\ &= -\frac{1}{4e^2} F^a_{\mu\nu} F^{b \; \mu\nu} h_{ab} + \mathcal{L}(\Phi, D_\mu\Phi) \quad , \end{aligned} \tag{12}$$

where $\mathcal{F}_{\mu\nu} = F^a_{\mu\nu} T_a$. The coupling constant e introduced in the first term is like the electric charge for the case of $G = U(1)$. We have not included a term of type (ii) in the Lagrangian since it is a total derivative. However, we will later find the need to include such a term from purely topological considerations.