DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH1020 Physics II — Problem Set 1 — November 2013

(Will be discussed during January 2014)

PH1020 involves discussions of scalar and vector fields such as the electric potential, the electric and magnetic fields. So we will be applying lots of ideas that were introduced in PH1010. This problem set is to let you recap and revisit concepts introduced in PH1010 in preparation for PH1020.

Coordinate Systems and Scalar fields

The use of coordinate systems that suit the problem at hand is often used in science and engineering problems. Let us use spherical polar coordinates to let you recollect various concepts. The spherical polar coordinates (r, θ, φ) are defined by the coordinate transformation

$$\begin{aligned} x &= r \sin\theta \cos\varphi \ , \\ y &= \sin\theta \sin\varphi \ , \\ z &= r\cos\theta \ . \end{aligned}$$

1. Determine the ranges of the various coordinates and identify if and when the coordinate transformation becomes singular.

2. Define $\hat{e}_r$ (resp. $\hat{e}_\theta$, $\hat{e}_\varphi$) to be the unit vector directed along increasing r (resp. θ, φ). Show that the *line element* takes the form
$$\delta\mathbf{r} = \delta x\ \hat{e}_x + \delta y\ \hat{e}_y + \delta z\ \hat{e}_z = h_r \delta r\ \hat{e}_r + h_\theta \delta\theta\ \hat{e}_\tau + h_\varphi \delta\varphi\ \hat{e}_\varphi \ ,$$
with the "scale factors" given by $h_r = 1$, $h_\theta = r$ and $h_\varphi = r\sin\theta$.

3. Further, verify that the unit vectors $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi)$ forming a right-handed basis (triad) of orthonormal vectors.

4. Hence, show that the kinetic energy of a particle of mass m in this coordinate system is given by
$$T = \frac{m}{2}\left(h_r^2\ \dot{r}^2 + h_\theta^2\ \dot{\theta}^2 + h_\varphi^2 \dot{\varphi}^2\right) \ .$$

5. Let $\psi(r, \theta, \varphi)$ be a smooth[1] scalar field. Then using
$$\psi(r + \delta r, \theta + \delta\theta, \varphi + \delta\varphi) - \psi(r, \theta, \varphi) = \nabla\psi \cdot \delta\mathbf{r},$$
determine the differential operator that represents ∇ in this coordinate system.

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[1]Smoothness implies that the scalar field and its first derivatives do not blow up in the region of interest. We will additionally assume that smoothness implies that the second derivative behaves nicely as well.

A vector field is smooth if none of its components or its first derivatives blow up. So care must be taken in ensuring smoothness before using the Stokes' and Gauss divergence theorems. Otherwise, we can end up with wrong conclusions analogous to the fallacious proofs of $2 = 1$ from high school.

6. Consider the vector field $\mathbf{v} = \frac{(-y\hat{e}_x + x\hat{e}_y)}{x^2+y^2}$. This is smooth everywhere except on points that lie on the z-axis. Let γ be a path given by traversing a circle of radius R centred at the origin and lying in the xy-plane with counterclockwise orientation. Show that the circulation of $\hat{v}$ is

$$\int_\gamma \mathbf{v} \cdot d\mathbf{r} = 2\pi .$$

However, if we ignore the non-smoothness at the origin and compute $\nabla \times \mathbf{v}$ in an attempt to use Stokes' theorem, we get zero. The correct statement is that $\nabla \times \mathbf{v} = 0$ away from the z-axis. But a point on the z-axis necessarily is part of the domain of integration. It is wrong to assume that this never occurs in physical situation. The vector field that we have used describes vortices in fluid flow ($\mathbf{v}$ is the velocity vector field of the fluid) and magnetic flux tubes in superconductors.

7. Next, consider the vector field $\mathbf{v} = \frac{K\hat{e}_r}{r^2}$ where $r = \sqrt{x^2+y^2+z^2}$. This vector field is smooth everywhere except at the origin. Let us compute the flux of this vector field through a sphere S_R of radius R centred at the origin. The flux of the vector field is given by

$$\int_{S_R} \mathbf{v} \cdot \mathbf{ds} = 4\pi K \ .$$

If we computed $\nabla \cdot \mathbf{v}$ – we would see that it vanishes everywhere except for the origin where it is non-smooth. So a blind application of the Gauss divergence theorem will give zero. Of course, the origin is necessarily enclosed by the sphere S_R and hence we cannot ignore the non-smoothness there. You might already know that the electric field of a charge at the origin is given by such a vector field and so this non-smoothness in not unphysical.

Now consider a region of interest in the form of a 'ball' of volume V bounded by a surface S and that the vector field of interest, $\mathbf{v}(\mathbf{r})$, is smooth in that region. **Helmholtz's theorem** tells us that if we are given the circulation and flux of vector field for all paths and surfaces that lie inside V, we can **uniquely** determine $\mathbf{v}(\mathbf{r})$ given *nice* boundary conditions on S such as giving the normal component of $\mathbf{v}$ on S. Practically speaking, this is equivalent to giving $(\nabla \times \mathbf{v})$ and $\nabla \cdot \mathbf{v}$ in the region.

Given a smooth vector field in $\mathbb{R}^3$ that falls off to zero sufficiently fast at spatial infinity, one can decompose it (non-uniquely) as follows:

$$\mathbf{v}(\mathbf{r}) = \nabla\phi(\mathbf{r}) + \nabla \times \mathbf{A}(\mathbf{r}) \ .$$

This is known as the **Helmholtz** decomposition. There is some obvious non-uniqueness associated with this decomposition that may be fixed by imposing conditions on ϕ and $\mathbf{A}$ at spatial infinity. Given $\nabla \cdot \mathbf{v}$, we see that it determines ϕ through the Poisson equation[2] $\nabla^2\phi = \nabla \cdot \mathbf{v}$ and similarly, one determines $\mathbf{A}$ by solving $\nabla^2\mathbf{A} = \nabla \times \mathbf{v}$ if one is given $\nabla \times \mathbf{v}$.

Suppose $\mathbf{v}$ is *solenoidal* i.e., $\nabla \cdot \mathbf{v} = 0$, then ϕ satisfies $\nabla^2\phi = 0$. The velocity field of an incompressible fluid is an example of a solenoidal vector field – this follows from the equation of continuity

$$\frac{\partial\rho(\mathbf{r},t)}{\partial t} + \nabla \cdot [\rho(\mathbf{r},t)\mathbf{v}(\mathbf{r},t)] = 0 \ ,$$

where $\rho(\mathbf{r},t)$ is the density of the fluid. For an incompressible fluid, the density is constant everywhere and time-independent. Thus, we can replace $\rho(\mathbf{r},t)$ by its constant value ρ_0 from which it follows that $\nabla \cdot \mathbf{v} = 0$.

8. The flow of an incompressible inviscid (non-viscous) fluid past an infinite cylinder (of radius R) whose axis is aligned with the z-axis is described by the vector field (in cylindrical polar coordinates (ϱ, φ, z))

$$\mathbf{v} = u(1 - \tfrac{R^2}{\varrho^2}) \sin\varphi \ \hat{e}_\varrho + u(1 + \tfrac{R^2}{\varrho^2}) \cos\varphi \ \hat{e}_\varphi \ , \qquad \text{for } \varrho = \sqrt{x^2 + y^2} \geq R \ .$$

   Verify that the vector field is irrotational and solenoidal. Obtain and intrepret the velocity field at $\varrho = R$ (i.e., on the surface of the cylinder) and $\varrho \to \infty$. Carry out the Helmholtz decomposition for this vector field. (Assume that $\phi \neq 0$, $\mathbf{A} = 0$ and set $\phi = (B\varrho + C/\varrho)\sin\varphi$ for some constants B and C to be determined by you.)
   Remark: Of course, real fluids are viscous. The flow past a cylinder of a real fluid is a classic fluid mechanics experiment which shows deviation from the above form for high velocities (more precisely, higher Reynolds number) and illustrates the route to turbulent flows. See for instance:
   http://labman.phys.utk.edu/phys221/modules/m9/turbulence.htm

In addition to the moodle page for this course, we will maintain a second page at http://sgovindarajan.wikidot.com/2014ph1020 where all problem sets, course resources, interesting links will be added as the course progresses.

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[2]There are standard methods to solve Poisson's and Laplace's equation and we might see some simple ones in PH1020. Given the enormous contribution of French scientists to modern science and engineering, it might be a good idea to learn to pronounce their names.☺More French names: Poincaré, Ampère.