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PH1020 Physics II **Problem Set 12** **21.4.2014**

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1. Let the displacement of a travelling wave be given by the function $u(\mathbf{r}\,,\,t) = u_0\,\exp\left[-(a^2y^2 - 2abty + b^2t^2)\right]$, where a and b are positive constants. What is its speed and direction of motion? Is this a harmonic wave? Schematically sketch u as a function of y for $t = 0$ and for some fixed positive t, and compare the two sketches.

2. The electric field of a plane EM wave in vacuum is given by

$$\mathbf{E} = E_0\,\hat{e}_x\,\cos\,kz\,\cos\,\omega t\,.$$

Find the corresponding magnetic field $\mathbf{H}$, given that $\mathbf{H} = 0$ at $t = 0$. Find the mean energy flux density of the wave.

3. Take two linearly polarized plane electromagnetic waves propagating in the z-direction, with their planes of polarization along the x and y directions respectively. The electric fields of the two waves have equal amplitudes, given by $|E_0|$. The frequency of each wave is ω, and its wave number is k.

   (a) Write down the electric and magnetic fields of the two waves.

   (b) Find the value of $\partial w/\partial t + \nabla\cdot\mathbf{S}$ for the plane waves, where w is the energy density and $\mathbf{S}$ is the Poynting vector.

   (c) Construct superpositions of the two waves to represent a wave that is (i) left circularly polarized, (ii) right circularly polarized. What are the electric and magnetic fields of these waves?

4. **Propagation of EM waves in a conducting medium**: This has been briefly touched upon in Problem Set 11, in connection with the skin effect. We go into this a little further here.

We know that in a conducting medium, Ohm's law says that the free current density is proportional to the electric field, $\mathbf{J}_f = \sigma\,\mathbf{E}$. Maxwell's equations for a linear medium then become

$$\nabla\cdot\mathbf{E} = \frac{\rho_f}{\epsilon}\,,\quad \nabla\cdot\mathbf{B} = 0\,,\quad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}\,,\quad \nabla\times\mathbf{B} = \mu\,\sigma\,\mathbf{E} + \mu\,\epsilon\,\frac{\partial\mathbf{E}}{\partial t}\,.$$

This would be a self-contained set of equations for $\mathbf{E}$ and $\mathbf{B}$, but for the presence of ρ_f in the first equation. However, using Ohm's law in the continuity equation for the free charge and current density, we get $\partial\rho_f/\partial t + (\sigma/\epsilon)\,\rho_f = 0$. This shows clearly that the transients in the free charge distribution die out in a time of the order of the **relaxation time** $\tau = \epsilon/\sigma$. Once the transients have died out, ρ_f vanishes, and the set of equations above becomes a homogeneous set of equations for the fields:

$$\nabla\cdot\mathbf{E} = 0\,,\quad \nabla\cdot\mathbf{B} = 0\,,\quad \nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t\,,\quad \nabla\times\mathbf{B} = \mu\,\sigma\,\mathbf{E} + \mu\,\epsilon\,\partial\mathbf{E}/\partial t\,.$$

Note that the only difference between this set and that for an insulating medium is the presence of the term $\mu\,\sigma\,\mathbf{E}$ in the RHS of the equation for curl $\mathbf{B}$ – for an **insulator**, the conductivity σ is zero. A (perfect) insulator is also called a "**non-lossy**" medium. A conducting medium (for which σ is non-zero) is also called a "**lossy**" medium, because the flow of a current will inevitably be accompanied by attenuation, as we shall see below.

Eliminating one of the two fields in the Maxwell equations above gives the wave equation for the other field in a conducting medium. We get

$$\mu\,\epsilon\,\frac{\partial^2\,\mathbf{E}}{\partial t^2} + \mu\,\sigma\,\frac{\partial\,\mathbf{E}}{\partial t} - \nabla^2\,\mathbf{E} = 0\,.$$

B also obeys exactlty the same equation. The presence of the first order time derivative implies that the waves will be attenuated: the amplitude will reduce to $1/e$ of its value at the surface of the medium at a certain **skin depth** d.

(a) Assume plane wave solutions of wave number k and (angular) frequency ω for **E** and **B**. On inserting these in the wave equations above, you will find that the wavenumber is necessarily complex in this case, implying attenuation. Show that the reciprocal of the skin depth is given by the expression

$$\frac{1}{d} = \omega\left(\frac{\epsilon\,\mu}{2}\right)^{1/2}\left[\sqrt{\left\{1 + \left(\frac{\sigma}{\epsilon\,\omega}\right)^2\right\}} - 1\right]^{1/2}.$$

Sketch the variation of d as a function of ω, assuming that all the other quantities in the formula above are constants.

(b) Show that the wavenumber of the EM waves is given by the expression

$$\frac{2\pi}{\lambda} = \omega\left(\frac{\epsilon\,\mu}{2}\right)^{1/2}\left[\sqrt{\left\{1 + \left(\frac{\sigma}{\epsilon\,\omega}\right)^2\right\}} + 1\right]^{1/2}.$$

The relation between the wavenumber and the frequency (the dispersion relation) is therefore quite complicated, in general. The (magnitude of the) wave velocity is given by ω divided by the wavenumber. (We must also distinguish between the wave velocity and the group velocity.) We do not go into further detail here, except to mention that these questions are studied in depth in connection with the propagation of EM waves in **wave guides**.

(c) We distinguish between **poor conductors** and **good conductors**, depending on whether $\sigma \ll \omega\,\epsilon$ or $\sigma \gg \omega\,\epsilon$. It is therefore clear that this distinction depends on the actual frequency of the EM waves whose propagation is under consideration. (Matters are further complicated in reality by the fact that, in general, σ itself is not a constant for a given medium, but is in fact a function of the frequency, i.e., $\sigma = \sigma(\omega)$. The function $\sigma(\omega)$ can be quite complicated, depending on the nature of the medium and the range of frequencies under consideration. We do not consider this aspect here.) Show that we get

$$d \approx \begin{cases} (2/\sigma)\,\sqrt{\epsilon/\mu} & \text{for poor conductors,} \\ \sqrt{2/(\omega\,\sigma\,\mu)} & \text{for good conductors.} \end{cases}$$

(d) The amplitude E_0 of the electric field of the EM wave is related to the amplitude H_0 of the magnetic field through $E_0 = \eta\,H_0$, where η is called the **characteristic impedance** of the medium. Show that

$$\eta = \sqrt{\frac{\mu}{\epsilon}}\left\{1 + \left(\frac{\sigma}{\epsilon\,\omega}\right)^2\right\}^{-1/4}.$$

What is the leading behaviour of η for poor conductors and good conductors, respectively?

Have a nice summer. The PH1020 team wishes you best of luck for the rest of your stay at IITM.