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PH1020 Physics II **Problem Set 4** **25.1.2014**

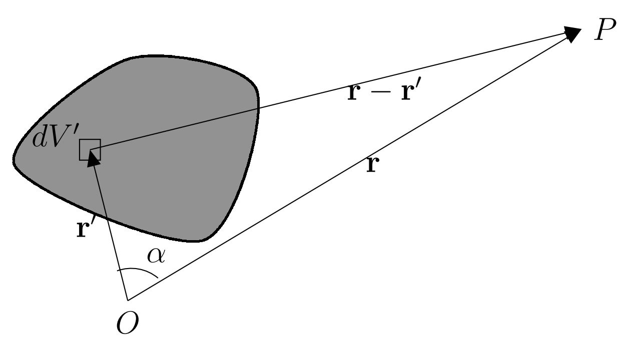
Multipole Expansion

This set is concerned with the multipole expansion of the electrostatic (or scalar) potential. In particular, we emphasize the *quadrupole term* in the expansion as you may have already encountered the monopole and dipole terms in some form. Please read the material below and work out the problems that follow.

As shown in class, consider a static charge distribution spread over a volume V with a density $\rho(\mathbf{r}')$ (we will use $\mathbf{r}'$ for the running coordinate of the points of the charge distribution). Then the electrostatic potential $\phi(\mathbf{r})$ at an arbitrary point, P, (see figure) with position vector $\mathbf{r}$, due to the charge distribution, is given by

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')\ dV'}{|\mathbf{r}-\mathbf{r}'|} \quad .$$

When $r \gg r'$ for all r' in V, we may expand $|\mathbf{r}-\mathbf{r}'|^{-1}$ in powers of r'/r to get



$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left[\frac{1}{r}\int_V \rho(\mathbf{r}')dV' + \frac{1}{r^2}\int_V r'\cos\alpha\ \rho(\mathbf{r}')\ dV' + \frac{1}{r^3}\int_V r'^2 \left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(\mathbf{r}')\ dV' + \cdots \right]$$

Recall that α is the angle between $\mathbf{r}$ and $\mathbf{r}'$ (it therefore changes from point to point as $\mathbf{r}'$ takes on different values in V). Since $q_{\text{tot}} = \int_V \rho(\mathbf{r}')dV'$ is the total charge in the distribution, and $\mathbf{p} = \int_V \rho(\mathbf{r}')\mathbf{r}'dV'$ is its dipole moment, we can write the first two terms in the expansion as

$$\boxed{\phi_{\text{monopole}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q_{\text{tot}}}{r}} \quad \text{and} \quad \boxed{\phi_{\text{dipole}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{e}_r}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}} \qquad (1)$$

respectively. We would like to write the next term (the quadrupole term) in a similar fashion. Note that both q and $\mathbf{p}$ *are intrinsic properties of the charge distribution itself* (they involve integrations over $\mathbf{r}'$, the running variable), and do *not* depend on the observation or field point $\mathbf{r}$ at which the potential is being determined. We would like to have the quadrupole term expressed similarly in terms of some property of the charge distribution. The relevant quantity is called the *quadrupole moment* of the distribution.

Writing $\mathbf{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$ and $\mathbf{r}' = x'\hat{e}_x + y'\hat{e}_y + z'\hat{e}_z$, the integrand in the third term can be written as

$$\frac{r'^2}{r^3}\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right) = \frac{3r'^2 r^2\cos^2\alpha - r'^2 r^2}{2r^5}$$
$$= \frac{3(xx' + yy' + zz')^2 - (x'^2 + y'^2 + z'^2)(x^2 + y^2 + z^2)}{2r^5} \quad .$$

The numerator is simplified as follows. Suppose we write the (components of) the position vector $\mathbf{r}$ in the form a column matrix, $\mathbf{r} = \left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right)$. The transpose of this is the row matrix: $(x\ y\ z)$. Then the numerator can be written as the product of the three matrices below, namely,

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 2x'^2 - y'^2 - z'^2 & 3x'y' & 3x'z' \\ 3y'x' & 2y'^2 - z'^2 - x'^2 & 3y'z' \\ 3z'x' & 3z'y' & 2z'^2 - x'^2 - y'^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Therefore, the quadrupole term can be written as

$$\boxed{\phi_{\text{quadrupole}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{r}\cdot \overleftrightarrow{Q} \cdot\mathbf{r}}{2r^5} \quad \text{or} \quad \frac{1}{4\pi\epsilon_0}\frac{x_i\ Q_{ij}\ x_j}{2r^5}\ ,} \tag{2}$$

where the symbol $\overleftrightarrow{Q}$ is called the *quadrupole moment.* It can also be written as a second-rank tensor[1] Q_{ij} with nine components given by

$$\begin{aligned}
Q_{xx} &= \int_V \rho(\mathbf{r}')\ (2x'^2 - y'^2 - z'^2)\ dV' \\
Q_{yy} &= \int_V \rho(\mathbf{r}')\ (2y'^2 - z'^2 - x'^2)\ dV' \\
Q_{zz} &= \int_V \rho(\mathbf{r}')\ (2z'^2 - x'^2 - y'^2)\ dV' \\
Q_{xy} = Q_{yx} &= \int_V \rho(\mathbf{r}')\ (3x'y')\ dV' \\
Q_{yz} = Q_{zy} &= \int_V \rho(\mathbf{r}')\ (3y'z')\ dV' \\
Q_{zx} = Q_{xz} &= \int_V \rho(\mathbf{r}')\ (3z'x')\ dV'\ .
\end{aligned}$$

[1]Recall that a second-rank tensor transforms under rotations as $Q'_{i_1 i_2} = R_{i_1 j_1} R_{i_2 j_2}\ Q_{j_1 j_2}$ where R_{ij} is the rotation matrix and a repeated index implies summation.

The monopole (charge) q is a scalar; the dipole moment $\mathbf{p}$ is a vector, with 3 components. The quadrupole moment $\overleftrightarrow{Q}$ is a tensor, with 9 components. However, since $Q_{xy} = Q_{yx}$, $Q_{yz} = Q_{zy}$, $Q_{zx} = Q_{xz}$ (symmetric) and further, $Q_{xx} + Q_{yy} + Q_{zz} = 0$ (traceless), $\overleftrightarrow{Q}$ only has 5 *independent* components. The quadrupole moment can also be written in the form of a real, symmetric (3×3) matrix with its trace equal to zero,

$$\begin{pmatrix} Q_{xx} & Q_{xy} & Q_{xz} \\ Q_{xy} & Q_{yy} & Q_{yz} \\ Q_{xz} & Q_{yz} & Q_{zz} \end{pmatrix} .$$

For a discrete distribution of n point charges q_i at positions $\mathbf{r}'_i$, the formulae above for the quadrupole moment are modified in obvious manner to

$$Q_{xx} = \sum_{i=1}^{n} q_i(2x_i'^2 - y_i'^2 - z_i'^2)$$

$$Q_{yy} = \sum_{i=1}^{n} q_i(2y_i'^2 - z_i'^2 - x_i'^2)$$

$$Q_{zz} = \sum_{i=1}^{n} q_i(2z_i'^2 - x_i'^2 - y_i'^2)$$

$$Q_{xy} = Q_{yx} = \sum_{i=1}^{n} q_i(3x_i'y_i') \ , \quad Q_{yz} = Q_{zy} = \sum_{i=1}^{n} q_i(3y_i'z_i')$$

$$Q_{zx} = Q_{xz} = \sum_{i=1}^{n} q_i(3z_i'x_i') \ .$$

The quadrupole moment (of charge and/or mass distributions) makes its appearance in several branches of physics such as nuclear physics, solid state physics, plasma physics, geophysics and astrophysics, in different contexts.

Now work out the problems given below.

1. A *point* dipole moment $\mathbf{p}$ is defined as follows: the separation $2a$ between the positive charge q and the negative charge $-q$ tends to zero, while q tends to infinity, such that $p \equiv 2aq$ is a finite positive number.

   (a) Show that the electric field at any point $\mathbf{r}$ due to a point dipole at the origin is given by

   $$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{3(\mathbf{p} \cdot \hat{e}_r)\hat{e}_r - \mathbf{p}}{r^3} \quad .$$

   (b) Now suppose the direction of the dipole moment $\mathbf{p}$ makes an angle θ_p with the z-axis. Calculate the electric flux through the upper hemisphere of a spherical surface of radius R centred at the origin.

2. (a) Show that if the total charge of a system is zero, the dipole moment does not depend on the location of the origin of the coordinates.

   (b) If a charge distribution has a non-zero monopole moment, show that it is possible to find an origin about which the dipole moment is zero.

   (c) Show that the quadrupole moment $\overleftrightarrow{Q}$ of a charge distribution does not depend on the location of the origin, if the monopole and dipole moments of the charge distribution are both zero.

3. Show that for a spherically symmetric charge distribution, the dipole, quadrupole and all higher moments about the centre of the distribution are identically zero.

4. Three point charges $+q$, $+2q$ and $-4q$ are held fixed at points with Cartesian coordinates $(0, 0, d)$, $(-d, 0, d)$ and (d, d, d) respectively, in a Cartesian coordinate system. Calculate the quadrupole moment of the charge distribution about that particular point about which the dipole moment vanishes.

5. A thin square non-conducting sheet of side length L has a uniform surface charge density σ. Let P be a point at a distance of $10L$ along the axis perpendicular to the plane of the sheet, and passing through its geometrical centre. Find the contributions to the electrostatic potential ϕ at P due to the charge on the sheet, from the monopole, dipole and quadrupole terms.

6. Find the monopole, dipole and quadrupole moments about the origin for the following charge distributions:

   (a) A line charge of constant charge density λ lying in the first quadrant of the xy-plane with one end at the origin, and making an angle φ_0 with the positive x-axis.

   (b) A solid cylinder of height H and radius R with its geometrical centre at the origin, having a uniform volume charge density ρ_0, and its axis along the z-axis.