DEPARTMENT OF PHYSICS
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PH5020 Electromagnetic Theory — Problem Set 10 — 13 Aoril. 2016

(Will be discussed on **April 15, 2016**)

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1. We have seen that electric and magnetic field mix to first-order in the relative velocity, $\mathbf{u}$, of two mutually inertial observers as follows:

$$\begin{aligned}\mathbf{E}'(\mathbf{x}',t') &= \mathbf{E}(\mathbf{x},t) + \frac{\mathbf{u}}{c}\times\mathbf{B}(\mathbf{x},t) + \mathcal{O}(u^2)\ ,\\ \mathbf{B}'(\mathbf{x}',t') &= \mathbf{B}(\mathbf{x},t) - \frac{\mathbf{u}}{c}\times\mathbf{E}(\mathbf{x},t) + \mathcal{O}(u^2)\ .\end{aligned}$$

Note that the above formula is in Gaussian units.

(a) Given a vector $\mathbf{u}$, we can decompose any vector $\mathbf{v} = \mathbf{v}_\perp + v_{||}\,\hat{u}$ in a vector perpendicular to $\mathbf{u}$ and parallel $\mathbf{u}$. Show that the above two equations can be rewritten as four equations:

$$\begin{aligned}\mathbf{E}'(\mathbf{x}',t')_\perp &= \mathbf{E}(\mathbf{x},t)_\perp + \frac{\mathbf{u}}{c}\times\mathbf{B}(\mathbf{x},t)_\perp + \mathcal{O}(u^2)\ ,\\ \mathbf{B}'(\mathbf{x}',t')_\perp &= \mathbf{B}(\mathbf{x},t)_\perp - \frac{\mathbf{u}}{c}\times\mathbf{E}(\mathbf{x},t)_\perp + \mathcal{O}(u^2)\ ,\\ E'(\mathbf{x}',t')_{||} &= E(\mathbf{x},t)_{||} + \mathcal{O}(u^2)\ ,\\ B'(\mathbf{x}',t')_{||} &= B(\mathbf{x},t)_{||} + \mathcal{O}(u^2)\ .\end{aligned}$$

(b) Extending arguments used to determine the linear piece, argue that the only possible terms at $\mathcal{O}(u^2)$ u^2 must be

$$\begin{aligned}\mathbf{E}'(\mathbf{x}',t')_\perp &= (1 + a_1\tfrac{u^2}{c^2})\ \mathbf{E}(\mathbf{x},t)_\perp + \frac{\mathbf{u}}{c}\times\mathbf{B}(\mathbf{x},t)_\perp + \mathcal{O}(u^3)\ ,\\ \mathbf{B}'(\mathbf{x}',t')_\perp &= (1 + b_1\tfrac{u^2}{c^2})\ \mathbf{B}(\mathbf{x},t)_\perp - \frac{\mathbf{u}}{c}\times\mathbf{E}(\mathbf{x},t)_\perp + \mathcal{O}(u^3)\ ,\\ E'(\mathbf{x}',t')_{||} &= (1 + a_2\tfrac{u^2}{c^2})\ E(\mathbf{x},t)_{||} + \mathcal{O}(u^3)\ ,\\ B'(\mathbf{x}',t')_{||} &= (1 + b_2\tfrac{u^2}{c^2})\ B(\mathbf{x},t)_{||} + \mathcal{O}(u^3)\ .\end{aligned}$$

where (a_1, a_2, b_1, b_2) are constants.

(c) Given that the exact transformations is given by (with $\gamma = (1-u^2/c^2)^{-1/2}$)

$$\begin{aligned}E_\parallel{}' &= E_\parallel\ , \quad \mathbf{E}_\perp{}' = \gamma\,(\mathbf{E}_\perp + \frac{\mathbf{u}\times\mathbf{B}_\perp}{c})\\ B_\parallel{}' &= B_\parallel\ , \quad \mathbf{B}_\perp{}' = \gamma\left(\mathbf{B}_\perp - \frac{\mathbf{u}\times\mathbf{E}_\perp}{c}\right)\ ,\end{aligned}$$

determine the constants (a_1, a_2, b_1, b_2) introduced in the previous part.

2. An arbitrary spatial rotation can be written as a rotation by an angle θ about an axis (specified by a unit vector $\hat{n}$). This is only true in three dimensions. Why?. Let us call the corresponding rotation matrix $R(\hat{n},\theta)$. Obtain an explicit expression for this 3×3 matrix. (Please see reference at the end of the problem set.)

3. An arbitrary Galilean transformation $(t, \mathbf{x}) \to (t', \mathbf{x}')$ is given by (with $R^i{}_j$ a rotation matrix)

$$t' = t + b^0 \ ,$$
$$x'^{\,i} = R^i{}_j \ x^j + v^i \ t + b^i \ ,$$

with ten parameters $\left\{b^0, \ R^i{}_j, v^i, \ b^i\right\}$. Repeated indices are summed over following the Einstein summation convention.

   (a) Show that $\frac{d^2\mathbf{x}}{dt^2} = 0$ implies that $\frac{d^2\mathbf{x}'}{dt'^2} = 0$.

   (b) Consider two successive such transformations with parameters $\left\{b_1^0, \ (R_1)^i{}_j, v_1^i, \ b_1^i\right\}$ and $\left\{b_2^0, \ (R_2)^i{}_j, v_2^i, \ b_2^i\right\}$ and obtain the composition rule.

   (c) Verify that all the axioms of a group are satisfied. This group is called the *Galilean* group.

4. An arbitrary Lorentz boost with rapidity, ϕ, along a direction $\hat{n}$ can be represented by a 4×4 matrix $B(\hat{n}, \phi)$. Obtain an explicit expression for $B(\hat{n}, \phi)$. In the limit of small rapidity, show that one recovers a Galilean boost along the direction $\hat{n}$.

5. (a) Given a time-like separation, dx^μ, with $d\mathbf{x} \neq 0$ along the x-direction and vanishing along the other two spatial directions. Find the Lorentz boost to a new frame where $d\mathbf{x}' = 0$.

   (b) Given a space-like separation, dx^μ, with $dx^0 \neq 0$ along the x-direction and vanishing along the other two spatial directions. Find the Lorentz boost to a new frame where $dx'^0 = 0$.

Recommended reading: V. Balakrishnan, *How is a vector rotated?*, Resonance, Vol. **4**, No. 10, pp. 61-68 (1999). Also available at the URL:

`http://www.physics.iitm.ac.in/%7Elabs/dynamical/pedagogy/`