DEPARTMENT OF PHYSICS
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PH5020 Electromagnetic Theory   Problem Set 11   April 20, 2016
(Will be discussed on **April 25, 2016**)

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1. The Lagrangian for a relativistic particle of mass m and charge q is given by

$$L = -m\sqrt{1-\dot{\mathbf{x}}^2} - q\ \Phi(\mathbf{x},t) + q\ \dot{\mathbf{x}}\cdot\mathbf{A}(\mathbf{x},t)\ .$$

   (a) Show that the action can be written in a form that shows it to be *manifestly* Lorentz invariant.

   (b) Show that the Euler-Lagrange equations of motion are those of a relativistic particle subject to the Lorentz force.

2. It turns out the the quantities $(E^2 - B^2)$ and $\mathbf{E}\cdot\mathbf{B}$ are Lorentz invariant (in Gaussian units). We would like to show that they are Lorentz invariant by writing them in manifestly invariant form. There are two invariant ('isotropic') tensors under Lorentz transformations. One of them is the metric tensor $\eta^{\mu\nu}$ and the other ithe four-dimensional Levi-Civita tensor defined in the following way. Analogous to the three-dimensional one that we know, $\epsilon^{\mu\nu\rho\sigma}$ is totally anti-symmetric i.e., it changes sign under exchange of any pair of indices. Further, let $\epsilon^{0123} = +1$. This fixes all other non-zero values to be ± 1.

   (a) Show that under Lorentz trsnsformations given by $\Lambda^\mu{}_\nu$ (with $\det(\Lambda) = 1$), one has

$$\epsilon'^{\mu_1\mu_2\mu_3\mu_4} = \Lambda^{\mu_1}{}_{\nu_1}\ \Lambda^{\mu_2}{}_{\nu_2}\ \Lambda^{\mu_3}{}_{\nu_3}\ \Lambda^{\mu_4}{}_{\nu_4}\ \epsilon'^{\nu_1\nu_2\nu_3\nu_4} = \det(\Lambda)\ \epsilon^{\mu_1\mu_2\mu_3\mu_4}\ .$$

   (b) Show that $(E^2 - B^2) \propto F_{\mu\nu}F^{\mu\nu}$ and work out the proportionality constant.

   (c) Show that $\mathbf{E}\cdot\mathbf{B} \propto \epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ and work out the proportionality constant.

   (d) Let $G^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$, Show tha $G^{0i} \propto B_i$ and $G^{ij} \propto \epsilon_{ijk}E_k$ and work out the proportionality constants. Verify that $\partial_\mu G^{\mu\nu} = 0$ gives two of Maxwell's equations in free space – we called this the Bianchi identity in class.

3. This problem is mostly about the kinematics of energy conservation in physical processes. We shall begin with the decay of a particle of mass M and three-momentum $\mathbf{P}$ into n decay products with masses m_i, and three-momenta $\mathbf{p}_i$ with $i = 1, 2, \ldots, n$. In special relativity, the conservation of four-momentum is given by

$$P^\mu = \sum_{i=1}^{n} p_i^\mu \quad , \quad \mu = 0,1,2,3\ ,$$

   where $P^\mu = (E/c, \mathbf{P})$ is the four-momentum of the initial particle and $p_i^\mu = (E_i/c, \mathbf{p}_i)$ represents the four-momentum of the i-th decay product.

   (a) Express E in terms of $\mathbf{P}$ and the mass M. Show that the integration measure $\frac{d^3P}{(2\pi)^3 2E}$ is Lorentz invariant.

(b) **Two-body decays:** For two decay products, in the rest frame of the particle of mass M, show that

$$\frac{E_1}{c^2} = \frac{M^2 - m_2^2 + m_1^2}{2M} \quad ,$$

$$\frac{|\mathbf{p}_1|}{c} = \frac{|\mathbf{p}_2|}{c} = \frac{\left[(M^2 - (m_1 + m_2)^2)(M^2 - (m_1 - m_2)^2)\right]^{1/2}}{2M} \quad .$$

(c) **Three-body decays:** In the rest frame of the particle of mass M:

i. Show that the conservation of three-momentum implies that the three decay products **must** lie in a plane.

ii. Define the Lorentz scalars, $m_{ij}^2 := (p_i + p_j)^2/c^2$. Show that $m_{ij}^2 > (m_i + m_j)^2$. *Hint:* Work in the rest frame of one of the particles, say the i-th one and compute $(p_i + p_j)^2$.

iii. Also show that $m_{12}^2 + m_{23}^2 + m_{13}^2 = M^2 + m_1^2 + m_2^2 + m_3^2$.

— Fin —