DEPARTMENT OF PHYSICS
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PH5020 Electromagnetic Theory — Problem Set 3 — 29 Jan. 2016

(Will be discussed on **February 5, 2016**)

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1. Let $\Phi_i(\mathbf{x})$ $(i = 1, 2)$ be the potential due to charge density $\rho_i(\mathbf{x})$ in $\mathbb{R}^3$. The two charge densities are localised and hence both the potentials vanish at spatial infinity. Prove the **Reciprocity** theorem:

$$\int_{\mathbb{R}^3} dV\ \Phi_1(\mathbf{x})\ \rho_2(\mathbf{x}) = \int_{\mathbb{R}^3} dV\ \Phi_2(\mathbf{x})\ \rho_1(\mathbf{x})\ .$$

2. Using the method of Green functions, obtain the solution of the problem

$$\frac{d^2\Phi(\theta)}{d\theta^2} = -\frac{\rho(\theta)}{\epsilon_0}\ ,\ 0 < \theta < \pi$$

subjected to the boundary condition $\Phi(0) = A$ and $\Phi(\pi) = B$, where A and B are constants.

3. Consider a two dimensional rectangular region $0 \leq x \leq a,\ 0 \leq y \leq b$. Show that the Green function, which vanishes on the boundary, can be expressed as

$$G(\mathbf{r}, \mathbf{r}') = \frac{2}{a} \sum_{n=1}^{\infty} g_n(y, y') \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi x'}{a}\right)\ ,$$

where the function $g_n(y, y')$ satisfies the differential equation

$$\left(-\frac{\partial^2}{\partial y^2} + \frac{n^2\pi^2}{a^2}\right) g_n(y, y') = 4\pi\delta(y - y')\ ,$$

with the boundary condition $g_n(0, y') = g_n(b, y') = 0$. Solve the above differential equation to determine $g_n(y, y')$. [1]

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[1] You may find the following Fourier series expansion, for the Dirac δ-function in an interval $0 \leq x \leq a$, useful:

$$\delta(x - x') = \frac{2}{a} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi x'}{a}\right)\ .$$

4. The most general solution to Laplace's equation in the plane (in plane polar coordinates (ϱ, φ)) for $0 < \varrho < \infty$ is of the form:

$$\Phi(\varrho, \varphi) = a_0 + b_0 \ \ln \ \varrho + \sum_{n=1}^{\infty} a_n \varrho^n \sin(n\varphi + \alpha_n) + \sum_{n=1}^{\infty} b_n \varrho^{-n} \sin(n\varphi + \beta_n) \ .$$

(a) Verify the assertion that Laplace's equation holds for $0 < \varrho < \infty$.

(b) Obtain the condition(s) under which the solution holds for $\varrho = 0$ as well.

(c) Let $\Phi(\varrho, \varphi)$ be the solution to the two-dimensional potential problem in the interior of the cylinder, $0 \leq \varrho \leq b$ subject to the boundary condition, $\Phi(\varrho = b, \varphi) = U(\varphi)$. Find the coefficients formally in terms of $U(\varphi)$.

(d) Substitute the coefficients into the series and sum it to obtain the potential inside the cylinder in the form of Poisson's integral:

$$\phi(\varrho, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} \ U(\varphi') \ \frac{b^2 - \varrho^2}{b^2 + \varrho^2 - 2b\varrho\cos(\varphi' - \varphi)} d\varphi'$$

(e) What modification(s) is(are) necessary if the potential is desired in the region that lies to the exterior the cylinder?