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PH5020 Electromagnetic Theory Problem Set 5 Feb. 22, 2016
(Will be discussed on **Feb 29, 2016**)

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1. (a) Beginning with the ansatz

$$G(\mathbf{x},\mathbf{x}') = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} g_{\ell,m}(r,r')\ Y_\ell^{*m}(\theta',\varphi')Y_\ell^m(\theta,\varphi)$$

for the Green function for the Laplacian (in spherical polar coordinates for $\mathbb{R}^3$), obtain the differential equation satisfied by $g_{\ell,m}(r.r')$.

(b) Show that $g_{\ell,m}(r.r')$ is independent of m and solve the differential equation in the two regions, $r < r'$ and $r > r'$. Show that the solution to the discontinuity at $r = r'$ is satisfied by

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{4\pi}{(2\ell+1)}\left(\frac{r_<^\ell}{r_>^{\ell+1}}\right)\ Y_\ell^{*m}(\theta',\varphi')Y_\ell^m(\theta,\varphi)\ ,$$

where $r_< = \min(r,r')$ and $r_> = \max(r,r')$.

(c) Choosing $\mathbf{x}'$ to point along the positive z-axis, one has $\theta' = 0$ and θ is the now the angle between $\mathbf{x}$ and $\mathbf{x}'$. After renaming θ to γ, Show that the Green function given above simplifies to:

$$G(\mathbf{x},\mathbf{x}') = \sum_{\ell=0}^{\infty}\left(\frac{r_<^\ell}{r_>^{\ell+1}}\right)\ P_\ell(\cos\gamma)\ .$$

2. We saw in class that the potential due to a localised charge distribution with density $\rho(\mathbf{x})$ has the following form (for large values of $\mathbf{x}$ far from the charges):

$$\Phi(\mathbf{x}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{a_{\ell m}}{r^{\ell+1}}\ Y_\ell^m(\theta,\varphi)\ ,$$

where

$$a_{\ell m} = \frac{4\pi}{2\ell+1}\int dV\,\rho(\mathbf{x})\ Y_\ell^{*m}(\theta,\varphi)\ r^\ell\ .$$

Show that

$$a_{2,2} = \frac{1}{12}\sqrt{\frac{15}{2\pi}}\,(Q_{xx} - Q_{yy} - 2iQ_{xy})\ ,$$

$$a_{2,1} = -\frac{1}{13}\sqrt{\frac{15}{8\pi}}\,(Q_{xz} - iQ_{yz})\ ,$$

$$a_{2,0} = \frac{1}{12}\sqrt{\frac{5}{4\pi}}\ Q_{zz}\ .$$

Recall that the quadrupole moment of a charge distribution, $\rho(\mathbf{x})$ is defined by

$$Q_{ij} = \int dV \rho(\mathbf{x}) \Big(3x_i x_j - \tfrac{1}{3}\delta_{ij} r^2 \Big) \ .$$

3. (a) If a charge distribution has a non-zero monopole moment, show that it is possible to find an origin about which the dipole moment is zero.

   (b) Show that the quadrupole moment of a charge distribution does not depend on the location of the origin, if the monopole and dipole moments of the charge distribution are both zero.

4. Find the monopole, dipole and quadrupole moments about the origin for the following charge distributions:

   (a) A line charge of constant charge density λ lying in the first quadrant of the xy-plane with one end at the origin, and making an angle φ_0 with the positive x-axis.

   (b) A solid cylinder of height H and radius R with its geometrical centre at the origin, having a uniform volume charge density ρ_0, and its axis along the z-axis.

5. Prove the following identity for the vector field $\mathbf{P}(\mathbf{x})$,

$$\int_V dV \ \mathbf{P} = \int_{S=\partial V} \mathbf{r} \left(\mathbf{P} \cdot \mathbf{dS} \right) - \int_V dV \ \mathbf{r} \ (\nabla \cdot \mathbf{P}) \, .$$

   Hint: Take the dot product of the above identity with a constant vector, $\mathbf{a}$ and prove the identify that follows. Argue that the answer should hold for arbitrary $\mathbf{a}$ leading to the above result.