DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH5020 Electromagnetic Theory — Problem Set 8 — 17 March 2016

(Will be discussed on **March 21, 2016**)

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1. A particle of mass m and charge q moves under the influence of an electromagnetic field. The electric and magnetic fields (not necessarily time-independent) can be written in terms of the scalar potential $\Phi(\mathbf{x}, t)$ and vector potential $\mathbf{A}(\mathbf{x}, t)$ as follows:

$$\mathbf{E} = -\nabla\Phi - \partial\mathbf{A}/\partial t \ , \quad \text{and} \quad \mathbf{B} = \nabla\times\mathbf{A} \ .$$

   (a) Show that the Euler-Lagrange equations for the Lagrangian given below reproduces the Lorentz force law:

$$L = \frac{1}{2}m\ \dot{\mathbf{x}}^2 + q\ \dot{\mathbf{x}}\cdot\mathbf{A}(\mathbf{x}, t) - q\ \Phi(\mathbf{x}, t) \ .$$

   This is an example of a Lagrangian that is **not** of the form $T - V$ as it involves a velocity-dependent conservative force.

   (b) How does the Lagrangian (and hence the action) change when we choose another set of potentials that give rise to the **same** electric and magnetic fields. To be precise, consider $\mathbf{A}', \Phi'$) given as follows:

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\lambda \quad , \quad \Phi' = \Phi - \partial_t\lambda \ ,$$

   where $\lambda(\mathbf{x}, t)$ is a function of spatial and time coordinates. Comment on your result.

2. Let $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ be a solution to the full time-dependent Maxwell's equations in a source-free region.

   (a) Let

$$U = \frac{\epsilon_0}{2}\ E^2 + \frac{1}{2\mu_0}\ B^2 \ ,$$

   be the energy density in the EM fields. Show that ME imply

$$\frac{\partial U}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{S} = 0 \ ,$$

   where the Poynting vector, $\mathbf{S} = k(\mathbf{E}\times\mathbf{B})$, and k is a constant to be determined by you.

   (b) The electric field of a spherical electromagnetic wave is given in spherical polar coordinates by (for $r \neq 0$)

$$\begin{aligned}\mathbf{E}(\mathbf{r}, t) &= \text{Re}\left(E_0 \sin\theta \left[\frac{1}{kr} + \frac{i}{k^2r^2}\right] \exp\left(ikr - i\omega t\right)\right)\ \hat{e}_\varphi \ , \\ &= E_0\sin\theta\left(\frac{\cos(kr-\omega t)}{kr} - \frac{\sin(kr-\omega t)}{k^2r^2}\right)\ \hat{e}_\varphi \ .\end{aligned}$$

   where E_0 is a positive real constant and $k = \omega/c$. Using the ansatz $\mathbf{B}(\mathbf{x}, t) = \widehat{\mathbf{B}}(\mathbf{x})\ e^{-i\omega t}$, obtain the magnetic field corresponding to this electric field using one of the ME.

   (c) Hence, compute the Poynting vector $\mathbf{S}$ for this electromagnetic wave and compute its average over one time period.

3. A beam of protons has a circular cross-section. Each proton has a velocity $\mathbf{v}$, and the beam constitutes a current I. Find the direction and magnitude of the Poynting vector $\mathbf{S}$ outside the beam, at a distance r from the axis of the beam.