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PH5020 Electromagnetic Theory | Problem Set 9 | April 1, 2016

(Will be discussed on **April 4, 2016**)

Propagation of EM waves in a conducting medium:

We know that in a conducting medium, Ohm's law says that the free current density is proportional to the electric field, $\mathbf{J}_f = \sigma\,\mathbf{E}$. Maxwell's equations for a linear medium then become

$$\nabla\cdot\mathbf{E} = \frac{\rho_f}{\epsilon}\,, \quad \nabla\cdot\mathbf{B} = 0\,, \quad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}\,, \quad \nabla\times\mathbf{B} = \mu\,\sigma\,\mathbf{E} + \mu\,\epsilon\,\frac{\partial\mathbf{E}}{\partial t}\,.$$

This would be a self-contained set of equations for $\mathbf{E}$ and $\mathbf{B}$, but for the presence of ρ_f in the first equation. However, using Ohm's law in the continuity equation for the free charge and current density, we get $\partial\rho_f/\partial t + (\sigma/\epsilon)\,\rho_f = 0$. This shows clearly that the transients in the free charge distribution die out in a time of the order of the **relaxation time** $\tau = \epsilon/\sigma$. Once the transients have died out, ρ_f vanishes, and the set of equations above becomes a homogeneous set of equations for the fields:

$$\nabla\cdot\mathbf{E} = 0\,, \quad \nabla\cdot\mathbf{B} = 0\,, \quad \nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t\,, \quad \nabla\times\mathbf{B} = \mu\,\sigma\,\mathbf{E} + \mu\,\epsilon\,\partial\mathbf{E}/\partial t\,.$$

Note that the only difference between this set and that for an insulating medium is the presence of the term $\mu\,\sigma\,\mathbf{E}$ in the RHS of the equation for curl $\mathbf{B}$ – for an **insulator**, the conductivity σ is zero. A (perfect) insulator is also called a "**non-lossy**" medium. A conducting medium (for which σ is non-zero) is also called a "**lossy**" medium, because the flow of a current will inevitably be accompanied by attenuation, as we shall see below.

Eliminating one of the two fields in the Maxwell equations above gives the wave equation for the other field in a conducting medium. We get

$$\mu\,\epsilon\,\frac{\partial^2\mathbf{E}}{\partial t^2} + \mu\,\sigma\,\frac{\partial\,\mathbf{E}}{\partial t} - \nabla^2\,\mathbf{E} = 0\,.$$

$\mathbf{B}$ also obeys exactlty the same equation. The presence of the first order time derivative implies that the waves will be attenuated: the amplitude will reduce to $1/e$ of its value at the surface of the medium at a certain **skin depth** δ.

1. Assume plane wave solutions of wave number k and (angular) frequency ω for $\mathbf{E}$ and $\mathbf{B}$. Let

$$\mathbf{E} = \mathrm{Re}\left[E_0\ \hat{n}\ e^{i(\mathbf{k}\cdot x - \omega t)}\right]\ ,$$

where $\hat{n}\cdot\mathbf{k} = 0$. On inserting this in the wave equations above, show that

$$k^2 - \mu\epsilon\omega^2 - i\omega\mu\sigma = 0 \quad \text{or} \quad \boxed{k = \pm(\mu\epsilon\omega^2 + i\mu\sigma\omega)^{1/2}}\ .$$

2. In the absence of any dissipation(i.e., when $\sigma = 0$), we have the usual wave equation $\mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} - \nabla^2\vec{E} = 0$ which has the dispersion relation $\omega = vk$ where $v = 1/\sqrt{\mu\epsilon}$. That is, $k = \sqrt{\epsilon\mu}\ \omega$. Therefore, it is the positive square root that is physical. Hence, we have

$$k = +(\mu\epsilon\omega^2 + i\mu\sigma\omega)^{1/2}\ . \tag{1}$$

Thus, k is necessarily complex implying attenuation. Show that the reciprocal of the skin depth is given by the expression

$$\frac{1}{\delta} = \omega\left(\frac{\epsilon\,\mu}{2}\right)^{1/2}\left[\sqrt{\left\{1 + \left(\frac{\sigma}{\epsilon\,\omega}\right)^2\right\}} - 1\right]^{1/2}\ .$$

3. Show that the wavenumber of the EM waves is given by the expression

$$\mathrm{Re}(k) = \frac{2\pi}{\lambda} = \omega \left(\frac{\epsilon\,\mu}{2}\right)^{1/2} \left[\sqrt{\left\{1 + \left(\frac{\sigma}{\epsilon\,\omega}\right)^2\right\}} + 1\right]^{1/2} .$$

The relation between the wavenumber and the frequency (the dispersion relation) is therefore quite complicated, in general. The (magnitude of the) wave velocity is given by ω divided by the wavenumber. (We must also distinguish between the wave velocity and the group velocity.) We do not go into further detail here, except to mention that these questions are studied in depth in connection with the propagation of EM waves in **wave guides**.

4. The amplitude E_0 of the electric field of the EM wave is related to the amplitude H_0 of the magnetic field through $E_0 = \eta\, H_0$, where η is called the **characteristic impedance** of the medium. Show that

$$\eta = \sqrt{\frac{\mu}{\epsilon}} \left\{1 + \left(\frac{\sigma}{\epsilon\,\omega}\right)^2\right\}^{-1/4} .$$

What is the leading behaviour of η for poor conductors and good conductors, respectively?

5. Consider a region in space where $z > 0$ is a conductor (with conductivity σ, permittivity ϵ and permeability μ as before) and $z < 0$ is vacuum. A plane electromagnetic wave whose electric field is given below is incident normally on the metal.

$$\mathbf{E}_{\text{incident}} = \mathrm{Re}\left[E_0\ \hat{e}_x\ e^{ikz - i\omega t}\right] .$$

The total electric field will then take the form (For simplicity, we have not indicated that the real part has to be taken but it is implicitly assumed to be the case.)

$$\mathbf{E}(\mathbf{x}, t) = \begin{cases} E_0\ \hat{e}_x\ e^{ikz - i\omega t} + E_r\ \hat{e}_x\ e^{-ikz - i\omega t} & z < 0\ , \\ E_t\ \hat{e}_x\ e^{ik'z - i\omega t} & z > 0\ . \end{cases}$$

where $k' = k_R + ik_I = +(\mu\epsilon\omega^2 + i\mu\sigma\omega)^{1/2}$ is the Show that

$$\mathbf{B}(\mathbf{x}, t) = \begin{cases} \frac{1}{c} E_0\ \hat{e}_y\ e^{ikz - i\omega t} - \frac{1}{c} E_r\ \hat{e}_y\ e^{-ikz - i\omega t} & z < 0\ , \\ \frac{k'}{\omega}\ E_t\ \hat{e}_y\ e^{ik'z - i\omega t} & z > 0\ . \end{cases}$$

6. For simplicity, assume $\mu = \mu_0$. Obtain the analogue of Fresnel's equations for E_r and E_t by imposing continuity of the components of $\mathbf{E}$ and $\mathbf{H}$ parallel to the interface at $z = 0$. Hence show that

$$\frac{E_t}{E_0} = \frac{2k}{k' + k} \quad \text{and} \quad \frac{E_r}{E_0} = \frac{k - k'}{k + k'} .$$

Consider a good conductor ($\sigma \gg \epsilon\omega$) for which one has $k' \sim \sqrt{\frac{\mu_0\sigma\omega}{2}}(1 + i)$ and $|k/k'| \ll 1$. In this case the above equation can be re-written as

$$\frac{E_t}{E_0} \sim \frac{2k}{k'} \quad \text{and} \quad \frac{E_r}{E_0} \sim -1 + \frac{2k}{k'} .$$

Note that $|E_t/E_0| \ll 1$ and $|E_r/E_0| \sim 1$ for a good conductor. Recall that E_0, E_r and E_t are assumed to be complex and hence the equations makes sense. It is also easy to see that the reflected wave is out of phase with respect to the incident wave as k' is complex.

7. Assume that the interface has area A, Compute the time-average of the Poynting vector $\langle\mathbf{S}\rangle = \frac{1}{2}\mathrm{Re}\left(\langle\mathbf{E} \times \mathbf{H}^*\rangle\right)$ at $z = 0$ and show that it is along the postive z direction, $\langle\mathbf{S} \cdot \hat{e}_z\rangle$. Next, compute the integral of the time-averaged ohmic dissipation, $u(z) := \frac{1}{2}\mathrm{Re}\left(\langle\mathbf{J} \cdot \mathbf{E}^*\rangle\right)$ for $z > 0$. Thus, $(A\langle\mathbf{S}\rangle \cdot \hat{e}_z)$ is the energy flux into the conductor and $(A\int_0^\infty u(z)dz)$ is the energy dissipated due to Ohmic heating. Show that $(\langle\mathbf{S}\rangle \cdot \hat{e}_z) = \int_0^\infty u(z)dz$.