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PH5020 Electromagnetic Theory Problem Set 1 15 Jan. 2018
(Will be discussed on **January 19, 2018**)

Vector fields

A vector field is smooth if none of its components or its first derivatives blow up. So care must be taken in ensuring smoothness before using the Stokes' and Gauss divergence theorems. Otherwise, we can end up with wrong conclusions analogous to the fallacious proofs of $2 = 1$ you may have seen in high school.

1. Consider the vector field $\mathbf{v} = \frac{(-y\hat{e}_x + x\hat{e}_y)}{x^2+y^2}$. This is smooth everywhere except on points that lie on the z-axis. Let γ be a path given by traversing a circle of radius R centred at the origin and lying in the xy-plane with counterclockwise orientation. Show that the circulation of $\mathbf{v}$ is

$$\oint_\gamma \mathbf{v} \cdot d\mathbf{r} = 2\pi.$$

However, if we ignore the non-smoothness at the origin and compute $\nabla \times \mathbf{v}$ in an attempt to use Stokes' theorem, we get zero. The correct statement is that $\nabla \times \mathbf{v} = 0$ away from the z-axis. But a point on the z-axis necessarily is part of the domain of integration. It is wrong to assume that this never occurs in physical situation. The vector field that we have used describes vortices in fluid flow ($\mathbf{v}$ is the velocity vector field of the fluid) and magnetic flux tubes in superconductors. *Using the Dirac delta function, express $\nabla \times \mathbf{v}$ such that it is valid at the origin and Stokes' theorem can be made to work.*

2. Next, consider the vector field $\mathbf{v} = \frac{K\hat{e}_r}{r^2}$ where $r = \sqrt{x^2+y^2+z^2}$. This vector field is smooth everywhere except at the origin. Let us compute the flux of this vector field through a sphere S_R of radius R centred at the origin. The flux of the vector field is given by

$$\int_{S_R} \mathbf{v} \cdot \mathbf{ds} = 4\pi K \ .$$

If we computed $\nabla \cdot \mathbf{v}$ – we would see that it vanishes everywhere except for the origin where it is non-smooth. So a blind application of the Gauss divergence theorem will give zero. Of course, the origin is necessarily enclosed by the sphere S_R and hence we cannot ignore the non-smoothness there. You might already know that the electric field of a charge at the origin is given by such a vector field and so this non-smoothness is not unphysical. *Using the Dirac delta function, express $\nabla \cdot \mathbf{v}$ such that it is valid at the origin and the Gauss divergence theorem can be made to work.*

Now consider a region of interest in the form of a 'ball' of volume V bounded by a surface S and that the vector field of interest, $\mathbf{v}(\mathbf{r})$, is smooth in that region. **Helmholtz's theorem** tells us that if we are given the circulation and flux of vector field for all paths and surfaces that lie inside V, we can **uniquely** determine $\mathbf{v}(\mathbf{r})$ given *nice* boundary conditions on S such as giving the normal component of $\mathbf{v}$ on S. Practically speaking, this is equivalent to giving $(\nabla \times \mathbf{v})$ and $\nabla \cdot \mathbf{v}$ in the region.

Given a smooth vector field in $\mathbb{R}^3$ that falls off to zero sufficiently fast at spatial infinity, one can decompose it (non-uniquely) as follows:

$$\mathbf{v}(\mathbf{r}) = \nabla\phi(\mathbf{r}) + \nabla \times \mathbf{A}(\mathbf{r}) \ .$$

This is known as the **Helmholtz** decomposition. There is some obvious non-uniqueness associated with this decomposition that may be fixed by imposing conditions on ϕ and $\mathbf{A}$ at spatial infinity and requiring $\nabla \cdot \mathbf{A} = 0$. Given $\nabla \cdot \mathbf{v}$, we see that it determines ϕ through the Poisson equation[1] $\nabla^2 \phi = \nabla \cdot \mathbf{v}$ and similarly, one determines $\mathbf{A}$ by solving $-\nabla^2 \mathbf{A} = \nabla \times \mathbf{v}$ if one is given $\nabla \times \mathbf{v}$.

Suppose $\mathbf{v}$ is *solenoidal* i.e., $\nabla \cdot \mathbf{v} = 0$, then ϕ satisfies $\nabla^2 \phi = 0$. The velocity field of an incompressible fluid is an example of a solenoidal vector field – this follows from the equation of continuity

$$\frac{\partial \rho(\mathbf{r}, t)}{\partial t} + \nabla \cdot [\rho(\mathbf{r}, t)\mathbf{v}(\mathbf{r}, t)] = 0 \ ,$$

where $\rho(\mathbf{r}, t)$ is the density of the fluid. For an incompressible fluid, the density is constant everywhere and time-independent. Thus, we can replace $\rho(\mathbf{r}, t)$ by its constant value ρ_0 from which it follows that $\nabla \cdot \mathbf{v} = 0$.

3. The flow of an incompressible inviscid (non-viscous) fluid past an infinite cylinder (of radius R) whose axis is aligned with the z-axis is described by the vector field (in cylindrical polar coordinates (ϱ, φ, z))

   $$\mathbf{v} = u(1 - \tfrac{R^2}{\varrho^2}) \sin\varphi \ \hat{e}_\varrho + u(1 + \tfrac{R^2}{\varrho^2}) \cos\varphi \ \hat{e}_\varphi \ , \quad \text{for } \varrho = \sqrt{x^2 + y^2} \geq R \ .$$

   Verify that the vector field is irrotational and solenoidal. Obtain and intrepret the velocity field at $\varrho = R$ (i.e., on the surface of the cylinder) and $\varrho \to \infty$. Carry out the Helmholtz decomposition for this vector field. (Assume that $\phi \neq 0$, $\mathbf{A} = 0$ and set $\phi = (B\varrho + C/\varrho)\sin\varphi$ for some constants B and C to be determined by you.)
   Remark: Of course, real fluids are viscous. The flow past a cylinder of a real fluid is a classic fluid mechanics experiment which shows deviation from the above form for high velocities (more precisely, higher Reynolds number) and illustrates the route to turbulent flows. See Figures 1 and 2 in this page:
   `http://www.thermopedia.com/content/1216/?tid=104\&sn=1410`
   to see how things behave at larger velocities (Reynolds number).

4. Using Dirac delta functions in the appropriate coordinates, express the following charge distributions as three-dimensional charge densities $\rho(\mathbf{x})$.

   (a) In spherical polar coordinates, a charge Q uniformly distributed over a spherical shell of radius R.

   (b) In cylindrical polar coordinates, a charge λ per unit length uniformly distributed over a cylindrical surface of radius b.

   (c) In spherical polar coordinates, a charge Q spread uniformly over a flat circular disc of negligible thickness and radius R.

   (d) In cylindrical polar coordinates, a charge Q spread uniformly over a flat circular disc of negligible thickness and radius R.

In addition to the moodle page for this course, we will maintain a second page at `http://sgovindarajan.wikidot.com/emt2018` where all problem sets, course resources, interesting links will be added as the course progresses.

[1] There are standard methods to solve Poisson's and Laplace's equation and we will see them later in the course. Given the enormous contribution of French scientists to modern science and engineering, it might be a good idea to learn to pronounce their names.☺ More French names: Poincaré, Ampère.