

**DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS**

PH5020 Electromagnetic Theory

Problem Set 10

2 Apr 2018

(Will be discussed on **April 6, 2018**)

1. We have seen that electric and magnetic field mix to first-order in the relative velocity, \mathbf{u} , of two mutually inertial observers as follows:

$$\begin{aligned}\mathbf{E}'(\mathbf{x}', t') &= \mathbf{E}(\mathbf{x}, t) + \frac{\mathbf{u}}{c} \times \mathbf{B}(\mathbf{x}, t) + \mathcal{O}(u^2), \\ \mathbf{B}'(\mathbf{x}', t') &= \mathbf{B}(\mathbf{x}, t) - \frac{\mathbf{u}}{c} \times \mathbf{E}(\mathbf{x}, t) + \mathcal{O}(u^2).\end{aligned}$$

Note that the above formula is in Gaussian units.

- (a) Given a vector \mathbf{u} , we can decompose any vector $\mathbf{v} = \mathbf{v}_\perp + v_\parallel \hat{u}$ in a vector perpendicular to \mathbf{u} and parallel \mathbf{u} . Show that the above two equations can be rewritten as four equations:

$$\begin{aligned}\mathbf{E}'(\mathbf{x}', t')_\perp &= \mathbf{E}(\mathbf{x}, t)_\perp + \frac{\mathbf{u}}{c} \times \mathbf{B}(\mathbf{x}, t)_\perp + \mathcal{O}(u^2), \\ \mathbf{B}'(\mathbf{x}', t')_\perp &= \mathbf{B}(\mathbf{x}, t)_\perp - \frac{\mathbf{u}}{c} \times \mathbf{E}(\mathbf{x}, t)_\perp + \mathcal{O}(u^2), \\ E'(\mathbf{x}', t')_\parallel &= E(\mathbf{x}, t)_\parallel + \mathcal{O}(u^2), \\ B'(\mathbf{x}', t')_\parallel &= B(\mathbf{x}, t)_\parallel + \mathcal{O}(u^2).\end{aligned}$$

- (b) Extending arguments used to determine the linear piece, argue that the only possible terms at $\mathcal{O}(u^2)$ u^2 must be

$$\begin{aligned}\mathbf{E}'(\mathbf{x}', t')_\perp &= (1 + a_1 \frac{u^2}{c^2}) \mathbf{E}(\mathbf{x}, t)_\perp + \frac{\mathbf{u}}{c} \times \mathbf{B}(\mathbf{x}, t)_\perp + \mathcal{O}(u^3), \\ \mathbf{B}'(\mathbf{x}', t')_\perp &= (1 + b_1 \frac{u^2}{c^2}) \mathbf{B}(\mathbf{x}, t)_\perp - \frac{\mathbf{u}}{c} \times \mathbf{E}(\mathbf{x}, t)_\perp + \mathcal{O}(u^3), \\ E'(\mathbf{x}', t')_\parallel &= (1 + a_2 \frac{u^2}{c^2}) E(\mathbf{x}, t)_\parallel + \mathcal{O}(u^3), \\ B'(\mathbf{x}', t')_\parallel &= (1 + b_2 \frac{u^2}{c^2}) B(\mathbf{x}, t)_\parallel + \mathcal{O}(u^3).\end{aligned}$$

where (a_1, a_2, b_1, b_2) are constants.

- (c) Given that the exact transformations is given by (with $\gamma = (1 - u^2/c^2)^{-1/2}$)

$$\begin{aligned}E'_\parallel &= E_\parallel, & \mathbf{E}'_\perp &= \gamma \left(\mathbf{E}_\perp + \frac{\mathbf{u} \times \mathbf{B}_\perp}{c} \right) \\ B'_\parallel &= B_\parallel, & \mathbf{B}'_\perp &= \gamma \left(\mathbf{B}_\perp - \frac{\mathbf{u} \times \mathbf{E}_\perp}{c} \right),\end{aligned}$$

determine the constants (a_1, a_2, b_1, b_2) introduced in the previous part.

2. (a) An arbitrary spatial rotation can be written as a rotation by an angle θ about an axis (specified by a unit vector \hat{n}). This is only true in three dimensions. Why?. Let us call the corresponding rotation matrix $R(\hat{n}, \theta)$. Obtain an explicit expression for this 3×3 matrix. (Please see reference at the end of the problem set.)

- (b) Let A be an anti-symmetric 3×3 matrix. Define the exponential of a matrix by

$$e^A := I + \sum_{n=1}^{\infty} \frac{A^n}{n!} .$$

Show that $R = e^A$ satisfies $RR^T = I$ and $\det(R) = 1$. In other words, it is a rotation matrix. It is a mathematical theorem that **all** rotation matrices (discussed in the previous part) arise this way.

3. An arbitrary Lorentz boost with rapidity, ϕ , along a direction \hat{n} can be represented by a 4×4 matrix $B(\hat{n}, \phi)$. Obtain an explicit expression for $B(\hat{n}, \phi)$. In the limit of small rapidity, show that one recovers a Galilean boost along the direction \hat{n} .
4. (a) Given a time-like separation, dx^μ , with $dx \neq 0$ along the x -direction and vanishing along the other two spatial directions. Find the Lorentz boost to a new frame where $dx' = 0$.
- (b) Given a space-like separation, dx^μ , with $dx^1 \neq 0$ along the x -direction and vanishing along the other two spatial directions. Find the Lorentz boost to a new frame where $dx'^0 = 0$.

Recommended reading: V. Balakrishnan, *How is a vector rotated?*, Resonance, Vol. 4, No. 10, pp. 61-68 (1999). Also available at the URL:

<http://www.physics.iitm.ac.in/%7Elabs/dynamical/pedagogy/>