

**DEPARTMENT OF PHYSICS**  
**INDIAN INSTITUTE OF TECHNOLOGY, MADRAS**

PH5020 Electromagnetic Theory

Problem Set 3

24 Jan. 2018

(Will be discussed on **February 2, 2018**)

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1. (a) Let  $G_D(\mathbf{x}, \mathbf{x}')$  denote the Dirichlet Green function for the Laplacian. Prove the symmetry property,  $G_D(\mathbf{x}, \mathbf{x}') = G_D(\mathbf{x}', \mathbf{x})$  for the Dirichlet Green function by using Green's second identity for the two functions,  $\phi = G_D(\mathbf{x}, \mathbf{y})$  and  $\psi = G_D(\mathbf{x}', \mathbf{y})$ , where  $\mathbf{y}$  is the variable of integration.
- (b) Let  $\Phi_i(\mathbf{x})$  ( $i = 1, 2$ ) be the potential due to charge density  $\rho_i(\mathbf{x})$  in  $\mathbb{R}^3$ . The two charge densities are localised and hence both the potentials vanish at spatial infinity. Prove the **Reciprocity** theorem:

$$\int_{\mathbb{R}^3} dV \Phi_1(\mathbf{x}) \rho_2(\mathbf{x}) = \int_{\mathbb{R}^3} dV \Phi_2(\mathbf{x}) \rho_1(\mathbf{x}) .$$

2. Using the method of Green functions, obtain the solution of the problem

$$\frac{d^2\Phi(\theta)}{d\theta^2} = -\frac{\rho(\theta)}{\epsilon_0} , \quad 0 < \theta < \pi$$

subjected to the boundary condition  $\Phi(0) = A$  and  $\Phi(\pi) = B$ , where  $A$  and  $B$  are constants.

3. Consider a two dimensional rectangular region  $0 \leq x \leq a$ ,  $0 \leq y \leq b$ . Show that the Green function, which vanishes on the boundary, can be expressed as

$$G(\mathbf{r}, \mathbf{r}') = \frac{2}{a} \sum_{n=1}^{\infty} g_n(y, y') \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi x'}{a}\right) ,$$

where the function  $g_n(y, y')$  satisfies the differential equation

$$\left(-\frac{\partial^2}{\partial y^2} + \frac{n^2\pi^2}{a^2}\right) g_n(y, y') = 4\pi\delta(y - y') ,$$

with the boundary condition  $g_n(0, y') = g_n(b, y') = 0$ . Solve the above differential equation to determine  $g_n(y, y')$ .<sup>1</sup>

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<sup>1</sup>You may find the following Fourier series expansion, for the Dirac  $\delta$ -function in an interval  $0 \leq x \leq a$ , useful:

$$\delta(x - x') = \frac{2}{a} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi x'}{a}\right) .$$

4. The most general solution to Laplace's equation in the plane (in plane polar coordinates  $(\varrho, \varphi)$ ) for  $0 < \varrho < \infty$  is of the form:

$$\Phi(\varrho, \varphi) = a_0 + b_0 \ln \varrho + \sum_{n=1}^{\infty} a_n \varrho^n \sin(n\varphi + \alpha_n) + \sum_{n=1}^{\infty} b_n \varrho^{-n} \sin(n\varphi + \beta_n) .$$

- (a) Verify the assertion that Laplace's equation holds for  $0 < \varrho < \infty$ .
- (b) Obtain the condition(s) under which the solution holds for  $\varrho = 0$  as well.
- (c) Let  $\Phi(\varrho, \varphi)$  be the solution to the two-dimensional potential problem in the interior of the cylinder,  $0 \leq \varrho \leq b$  subject to the boundary condition,  $\Phi(\varrho = b, \varphi) = U(\varphi)$ . Find the coefficients formally in terms of  $U(\varphi)$ .
- (d) Substitute the coefficients into the series and sum it to obtain the potential inside the cylinder in the form of Poisson's integral:

$$\Phi(\varrho, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} U(\varphi') \frac{b^2 - \varrho^2}{b^2 + \varrho^2 - 2b\varrho \cos(\varphi' - \varphi)} d\varphi'$$

- (e) What modification(s) is(are) necessary if the potential is desired in the region that lies to the exterior the cylinder?

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