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PH5020 Electromagnetic Theory — Problem Set 4 — Feb 3, 2018

(Will be discussed on **Feb. 9 , 2018**)

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1. In polar coordinates in the plane, we denote the Green function for the Laplacian by $G(\varrho, \varphi; \varrho', \varphi')$. It satisfies

$$-\nabla^2 G(\varrho, \varphi; \varrho', \varphi') = 4\pi \frac{\delta(\varrho - \varrho')}{\varrho}\ \delta(\varphi - \varphi')\ .$$

Recall that the factor of $\frac{1}{\varrho}$ in the RHS cancels the ϱ in the integration measure $d^2x = \varrho d\varrho d\varphi$.

   (a) The completness relation for the orthonormal basis $\mathcal{B} := \big(\frac{1}{\sqrt{2\pi}}\ e^{im\varphi} | m \in \mathbb{Z}\big)$ for functions on the circle (with coordinate φ) is

$$\frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\varphi - \varphi')} = \delta(\varphi - \varphi')\ .$$

   Using this show explicitly that the Green function can be expressed as follows:

$$G(\varrho, \varphi; \varrho', \varphi') = \frac{1}{2\pi} \sum_{-\infty}^{\infty} e^{im(\varphi - \varphi')} g_m(\varrho, \varrho')$$

   where the radial Green function satisfies

$$\left(\frac{1}{\varrho} \frac{d}{d\rho} \left(\varrho \frac{dg_m}{d\varrho} \right) - \frac{m^2}{\varrho^2} g_m(\varrho, \varrho') \right) = -\frac{4\pi}{\varrho} \delta(\varrho - \varrho')\ .$$

   (b) By solving the equation separately for $\varrho < \varrho'$ and $\varrho > \varrho'$ and matching the solutions at $\varrho = \varrho'$, show that the free-space Green function takes the symmetric form:

$$G(\varrho, \varphi; \varrho', \varphi') = -\ln({\varrho_>}^2) + 2 \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\varrho_<}{\varrho_>} \right)^m \cos[m(\varphi - \varphi')]$$

   where $\varrho_<(\varrho_>)$ is the smaller (larger) of ϱ and ϱ'.

2. Now consider the situation where we replace the two-dimensional plane by the upper half-plane. In plane polar coordinates, this implies that $0 \leq \varphi \leq \pi$. We wish to study the Dirichlet problem where the potential is specified on the x-axis.

   (a) In order to obtain the Green function for this problem, we need to study restrict to functions that vanish at $\varphi = 0, \pi$. Modify the basis $\mathcal{B}$ given in problem 1 above to make it suitable for the problem here. Write the completeness relation for the new basis.

   (b) Show that the Green function in the region described in the above problem can be expressed as:

$$G(\mathbf{r}, \mathbf{r}') = \frac{2}{\pi} \sum_{m=1}^{\infty} g_m(\varrho, \varrho') \sin(m\varphi) \sin(m\varphi')\ \ ,$$

   where $g_m(\varrho, \varrho')$ satisfies the same differential equation as in problem 1.

3. Finally, let us consider the Dirichlet problem where the region is a wedge of angle β in the plane. In plane polar coordinates, this implies that $0 \leq \varphi \leq \beta$. Thus $\beta = \pi$ corresponds to problem 2 above.

   (a) Show that the Dirichlet Green function in the region described in the above problem can be expressed as:

   $$G(\mathbf{r}, \mathbf{r}') = \frac{2}{\beta} \sum_{m=1}^{\infty} g_m(\rho, \rho') \sin\left(\frac{m\pi\varphi}{\beta}\right) \sin\left(\frac{m\pi\varphi'}{\beta}\right) ,$$

   where $g_m(\varrho, \varrho')$ satisfies the differential equation:

   $$\left(\frac{1}{\varrho}\frac{d}{d\rho}\left(\varrho\frac{dg_m}{d\varrho}\right) - \frac{m^2\pi^2}{\beta^2\varrho^2} g_m(\varrho, \varrho')\right) = -\frac{4\pi}{\varrho}\delta(\varrho - \varrho') .$$

   (b) Show the the following expression for $g_n(\varrho, \varrho')$ is a solution to the differential equation

   $$g_n(\varrho, \varrho') = \frac{2\beta}{n} \left(\frac{\varrho_<}{\varrho_>}\right)^{n\pi/\beta} ,$$

   where $\varrho_< = \min(\varrho, \varrho')$ and $\varrho_> = \max(\varrho, \varrho')$.

   (c) Argue that setting $\beta = 2\pi$ is not the same as problem 1. (*Hint:* Study the basis that you get and see what is missing.)

   (d) Use the above solution to show that the Green function appropriate for the Dirichlet boundary condition for the two dimensional region bounded by the lines $\varphi = 0, \varphi = \beta$ and $\varrho = a$ is given by

   $$g_n(\varrho, \varrho') = \frac{2\beta}{n} \left(\frac{\varrho_<}{a}\right)^{n\pi/\beta} \left(\left(\frac{a}{\varrho_>}\right)^{n\pi/\beta} - \left(\frac{\varrho_>}{a}\right)^{n\pi/\beta}\right) .$$

   Again, check whether you recover the answer to part (b) in a suitable limit.