DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH5020 Electromagnetic Theory   Problem Set 5   Feb. 8, 2018
(Will be discussed on **Feb 16, 2018**)

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1. We saw in class that the potential due to a localised charge distribution with density $\rho(\mathbf{x})$ has the following form (for points $\mathbf{x}$ that are far from the charges):
$$\Phi(\mathbf{x}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{a_{\ell m}}{r^{\ell+1}} \, Y_\ell^m(\theta, \varphi) \ ,$$
where
$$a_{\ell m} = \frac{4\pi}{2\ell+1} \int dV \, \rho(\mathbf{x}) \, Y_\ell^{*m}(\theta, \varphi) \, r^\ell \ .$$
Work out the precise relationship between the dipole moment of the charge distribution and $a_{1,m}$ for $m = 0, \pm 1$. Show that
$$\begin{aligned} a_{2,2} &= \frac{1}{12}\sqrt{\frac{15}{2\pi}} \left(Q_{xx} - Q_{yy} - 2iQ_{xy}\right) \ , \\ a_{2,1} &= -\frac{1}{3}\sqrt{\frac{15}{8\pi}} \left(Q_{xz} - iQ_{yz}\right) \ , \\ a_{2,0} &= \frac{1}{2}\sqrt{\frac{5}{4\pi}} \, Q_{zz} \ . \end{aligned}$$
Recall that the quadrupole moment of a charge distribution, $\rho(\mathbf{x})$ is defined by
$$Q_{ij} = \int dV \rho(\mathbf{x}) \Big(3x_i x_j - \delta_{ij} r^2\Big) \ .$$

2. (a) If a charge distribution has a non-zero monopole moment, show that it is possible to find an origin about which the dipole moment is zero.

   (b) Show that the quadrupole moment of a charge distribution does not depend on the location of the origin, if the monopole and dipole moments of the charge distribution are both zero.

3. Find the monopole, dipole and quadrupole moments about the origin for the following charge distributions:

   (a) A line charge of constant charge density λ lying in the first quadrant of the xy-plane with one end at the origin, and making an angle φ_0 with the positive x-axis.

(b) A solid cylinder of height H and radius R with its geometrical centre at the origin, having a uniform volume charge density ρ_0, and its axis along the z-axis.

4. Let S_1 and S_2 be concentric spheres of radii R_1 and R_2 ($> R_1$). Show that the capacitance matrix is given by

$$C = C_0 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} , \tag{1}$$

where $C_0 := (4\pi\epsilon_0)\ \frac{R_1 R_2}{R_2 - R_1}$.

5. Show that an open capacitor made of the two surfaces S_1 and S_2 has capacitance $C = \frac{C_{11}C_{22} - C_{12}^2}{C_{11}+C_{22}+2C_{12}}$ where we have written the answer in terms of the coefficients of capacitance for the system of two surfaces.

6. Prove the following identity for the vector field $\mathbf{P}(\mathbf{x})$,

$$\int_V dV\ \mathbf{P} = \int_{S=\partial V} \mathbf{r}\,(\mathbf{P}\cdot\mathbf{dS}) - \int_V dV\ \mathbf{r}\ (\nabla\cdot\mathbf{P})\,.$$

Hint: Take the dot product of the above identity with a constant vector, $\mathbf{a}$ and prove the identify that follows. Argue that the answer should hold for arbitrary $\mathbf{a}$ leading to the above result.

7. **Additional question which will not be discussed in the tutorial.** The upper hemisphere of a sphere of radius R is kept at potential $V_0 > 0$ and the lower hemisphere is at potential $-V_0$. There is no charges elsewhere.

(a) Compute the potential in the interior of the sphere in two ways: (i) Use the Dirichlet Green function given by the method of images and (ii) Use the Dirichlet Green function derived in class in terms of spherical harmonics.

(b) What is the charge density on the surface of the sphere?

(c) What is the potential in the region outside the sphere?