DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH5020 Electromagnetic Theory | Problem Set 7 | Feb 28, 2018

(Will be discussed on **Mar 9, 2018**)

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1. Let $\mathbf{J}(\mathbf{x})$ be a vector field satisfying $\nabla \cdot \mathbf{J} = 0$ and let $\mathbf{a}$ be a constant vector. Also assume that $\mathbf{J}$ is localised and vanishing outside of the volume V,

   (a) Prove the following identities:

   $$\int_V dV\ \mathbf{J} = 0 \quad \text{and} \quad \int_V dV\ \left(x_i J_j + x_j J_i\right) = 0\ ,$$
   $$(\mathbf{a}\cdot\mathbf{x})\ \mathbf{J} = (\mathbf{a}\cdot\mathbf{J})\ \mathbf{x} - \mathbf{a}\times(\mathbf{x}\times\mathbf{J})\ .$$

   (b) Using the above identities, show that

   $$\int_V dV\ (\mathbf{a}\cdot\mathbf{x})\ \mathbf{J} = -\frac{1}{2}\mathbf{a}\times\int_V dV\ (\mathbf{x}\times\mathbf{J})\ .$$

   Remark: These identities were assumed in class during the calculation that shows that the dipole ($\ell = 1$) contribution to the vector potential due to a current density $\mathbf{J}(\mathbf{x})$ is given by

   $$\mathbf{A}_{\text{dipole}}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{e}_r}{r^2}\ ,$$

   where the magnetic dipole moment $\mathbf{m}$ of the current density

   $$\mathbf{m} = \frac{1}{2}\int_V d^3x\ \left(\mathbf{x}\times\mathbf{J}(\mathbf{x})\right)$$

2. (a) The electrostatic potential due to a point electric dipole with moment $\mathbf{p}$ (located at the origin) is given by

   $$\Phi_{\text{dipole}} = \frac{1}{4\pi\epsilon_0}\ \frac{\mathbf{p}\cdot\hat{e}_r}{r^2}\ ,$$

   Find the electric field, $\mathbf{E}$, due to the dipole for an arbitrary orientation of the dipole moment $\mathbf{p}$.
   Hint: Let the spherical polar coordinates of the vectors $\mathbf{r}$ and $\mathbf{p}$ be (r,θ,φ) and (p,θ_p,φ_p) respectively. Then the cosine of the angle between these two vectors is given by

   $$\cos\,\alpha = \cos\,\theta\,\cos\,\theta_p + \sin\,\theta\,\sin\,\theta_p\,\cos\left(\varphi-\varphi_0\right).$$

   (b) The vector potential due to magnetic dipole with moment, $\mathbf{m}$, (located at the origin) is given by

   $$\mathbf{A}_{\text{dipole}} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{e}_r}{r^2}\ .$$

   Find the magnetic field, $\mathbf{B}$, due to the magnetic dipole.

   (c) Compare and comment on the two answers.

3. Find the exact magnetic field a distance z above the centre of a square loop of side s lying in the xy-plane and carrying current I. Verify that it reduces to the field of a magnetic dipole, with the appropriate magnetic dipole moment, when $z \gg s$.

4. A flat circular disc of radius R lies in the xy-plane with its centre at the origin. Th disc has a uniform surface charge density $-\sigma$ from its centre up to a radius $R/\sqrt{2}$, and a uniform surface charge density $+\sigma$ for radii from $R/\sqrt{2}$ up to R. The disc rotates with a constant angular velocity ω about the z-axis. Show that the disc (whose total charge is zero) has a magnetic dipole moment directed along the z-axis, with magnitude equal to $\pi\sigma\omega R^4/8$.