DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH5020 Electromagnetic Theory | Problem Set 8 | March 12, 2018

(Will be discussed on **Mar 16, 2018**)

---

1. In a region with no free currents, one has $\nabla \times \mathbf{H} = 0$. In this case, in analogy with electrostatics, one defines the magnetic scalar potential as follows:

$$\mathbf{H}(\mathbf{x}) = -\nabla\Phi_M(\mathbf{x}) \ .$$

Consider a cylinder of length L and radius R carries a uniform magnetization $\mathbf{M}$ parallel to its its axis of symmetry. With no loss of generality, assume that the axis of symmetry is the z-axis.

   (a) Show that the magnetic scalar potential satisfies Laplace's equation.

   (b) Solve for the magnetic scalar potential in the interior and exterior of the cylinder by using the general form of the solution to Laplace's equation.

   (c) Compute the magnetic field $\mathbf{H}$ and the magnetic induction $\mathbf{B}$ from the scalar potential.

   (d) Compare and contrast the results with that of the similar problem in electrostatics (see problem set 6).

2. A cylinder of length L and radius R carries a magnetization $\mathbf{M} = k\varrho^2\hat{e}_\varphi$, where k is a constant, and the axis of the cylinder is the z-axis. Verify that $\nabla \cdot \mathbf{M} = 0$ and hence $\nabla \cdot \mathbf{H} = 0$. Thus, find the magnetic field inside and outside the cylinder using the method used in the previous problem.

3. (a) A localised charge distribution, $\rho(\mathbf{x})$ occupying a finite volume V is placed in a region of non-uniform (external) electric field. Show/argue that the electric force and torque are given by the formulae

$$\mathbf{F} = \int_V dV\ \rho(\mathbf{x})\mathbf{E}(\mathbf{x})$$
$$\mathbf{N} = \int_V dV\ \rho(\mathbf{x})\ \left(\mathbf{x} \times \mathbf{E}(\mathbf{x})\right)$$

   (b) Now consider the situation where the monopole moment of the distribution vanishes. By expanding the electric field as a Taylor series about a point O in the volume, show that the **leading** terms are

$$\begin{aligned}\mathbf{F} &= (\mathbf{p} \cdot \nabla)\, \mathbf{E}(\mathbf{x}) + \cdots \\ &= -\nabla\left(-\mathbf{p} \cdot \mathbf{E}\right) + \cdots \\ \mathbf{N} &= \mathbf{p} \times \mathbf{E}(\mathbf{x}) + \cdots\end{aligned}$$

   where we should set $\mathbf{x}$ to its value at O at the end. If the charge distribution is a point dipole, then O is taken to be the location of the point dipole.

4. (a) We will now repeat the discussion for a magnetic case. Let $\mathbf{J}((x))$ be a localised and steady current density occupying a finite volume V is placed in a region of non-uniform (external) magnetic field[1]. Show/argue that the magnetic force and torque are given by the formulae

$$\mathbf{F} = \int_V dV\ \mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x})$$
$$\mathbf{N} = \int_V dV\ \mathbf{x} \times \left(\mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x})\right)$$

[1] The sources of this external magnetic field are outside V and hence $\nabla \times \mathbf{B} = 0$ in V.

(b) By expanding the magnetic field as a Taylor series about a point O in the volume, show that the **leading** terms are

$$\begin{aligned}\mathbf{F} &= (\mathbf{m}\cdot\nabla)\,\mathbf{B}(\mathbf{x}) + \cdots \\ &= -\nabla\,(-\mathbf{m}\cdot\mathbf{B}) + \cdots \\ \mathbf{N} &= \mathbf{m}\times\mathbf{B}(\mathbf{x}) + \cdots\end{aligned}$$

where we should set $\mathbf{x}$ to its value at O at the end. If the charge distribution is a point dipole, then O is taken to be the location of the point dipole.

The last two problems illustrates the 'derivation' of formulae for the potential energy (and torque) due to an electric (resp. magnetic) dipole in inhomogeneous external electric (resp.magnetic) fields. These formulae should appear familiar in the context of uniform fields that you may have seen in earlier courses. The ignored terms indicated by an ellipsis involve higher derivatives of the electric/magnetic field.