DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH5020 Electromagnetic Theory — Problem Set 9 — March 17, 2018

(Will be discussed on **Mar 16, 2018**)

---

1. An ideal parallel plate capacitor of capacitance C has circular plates located at $z = 0$ and $z = d$ respectively. The axis of symmetry of the capacitor is the z-axis. The medium between the plates is a LIH dielectric with dielectric constant $\kappa = \epsilon/\epsilon_0$ and permeability μ_0. A voltage $V = V_0 \sin \omega t$ is applied across the plates of the capacitor. The frequency is assumed to small enough for the quasi-static limit to be applicable.

   (a) What is the electric displacement, $\mathbf{D}(\mathbf{r}, t)$, inside the capacitor?

   (b) Assuming that $\mathbf{H}(\mathbf{r}, t) = H(\varrho, t)\ \hat{e}_\varphi$, use Ampère's law to determine the induced auxiliary field due to the displacement current.

   (c) Obtain the expression for $\left|\frac{\partial B(\varrho,t)}{\partial \varrho}\right|$ at $\varrho = 0$ and $t = 0$.

   (d) Given that $\kappa = 1.05$, $d = 1.2$cm, $V_0 = 480$V and the angular frequency $\omega = 7850$rad/s, obtain an order of magnitude estimate for $\left|\frac{\partial B(\varrho,t)}{\partial \varrho}\right|$ at $\varrho = 0$ and $t = 0$. Express your answer in μG/cm, where 1G $= 10^{-4}$T. These are real values taken from the reference given below.

   Reference: D. F. Bartlett and T. R. Corle, "Measuring Maxwell's Displacement Current Inside a Capacitor", Phys. Rev. Lett. **55** (1985) 59. (available for download from the moodle page for the course)

2. An arbitrary Galilean transformation $(t, \mathbf{x}) \to (t', \mathbf{x}')$ is given by (with $R^i{}_j$ a rotation matrix)

$$t' = t + b^0 \ ,$$
$$x'^{\,i} = R^i{}_j\ x^j + v^i\ t + b^i \ ,$$

   with ten parameters $\left\{b^0,\ R^i{}_j, v^i,\ b^i\right\}$. Repeated indices are summed over following the Einstein summation convention.

   (a) Show that $\frac{d^2\mathbf{x}}{dt^2} = 0$ implies that $\frac{d^2\mathbf{x}'}{dt'^2} = 0$.

   (b) Consider two successive such transformations with parameters $\left\{b_1^0,\ (R_1)^i{}_j, v_1^i,\ b_1^i\right\}$ and $\left\{b_2^0,\ (R_2)^i{}_j, v_2^i,\ b_2^i\right\}$ and obtain the composition rule.

   (c) Verify that all the axioms of a group are satisfied. This group is called the *Galilean* group.

3. A particle of mass m and charge q moves under the influence of an electromagnetic field. The electric and magnetic fields (not necessarily time-independent) can be written in terms of the scalar potential $\Phi(\mathbf{x}, t)$ and vector potential $\mathbf{A}(\mathbf{x}, t)$ as follows:

$$\mathbf{E} = -\nabla\Phi - \partial\mathbf{A}/\partial t \ , \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A} \ .$$

   (a) Show that the Euler-Lagrange equations for the Lagrangian given below reproduces the Lorentz force law:

$$L = \frac{1}{2} m\ \dot{\mathbf{x}}^2 + q\ \dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}, t) - q\ \Phi(\mathbf{x}, t) \ .$$

   This is an example of a Lagrangian that is **not** of the form $T - V$ as it involves a velocity-dependent conservative force.

(b) How does the Lagrangian (and hence the action) change when we choose another set of potentials that give rise to the **same** electric and magnetic fields. To be precise, consider $(\mathbf{A}', \Phi')$ given as follows:

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\lambda \quad , \quad \Phi' = \Phi - \partial_t \lambda \ ,$$

where $\lambda(\mathbf{x}, t)$ is a function of spatial and time coordinates. Comment on your result.

4. (a) By carrying out the Legendre transform of the Lagrangian in the previous problem, obtain the following Hamiltonian

$$H = \frac{(\mathbf{p} - q\mathbf{A}(\mathbf{x}, t))^2}{2m} + q\Phi(\mathbf{x}, t) \ .$$

(b) Show that Hamilton's equation of motion are equivalent to the Lorentz force law.