Special classical solutions: Solitons

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The Lagrangian density for a single scalar field is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\ \partial^\mu\phi - U(\phi) \quad , \tag{1}$$

where $U(\phi)$ is the potential for the scalar field. Let us assume that $U(\phi)$ is bounded from below and that by suitably adding a constant, this bound is shifted to zero i.e., $U(\phi) \geq 0$ everywhere. The equation of motion is

$$\Box\,\phi = -\frac{\partial U(\phi)}{\partial\phi}$$

The stress tensor corresponding to the above Lagrangian density is

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L} \ .$$

The energy density $\mathcal{H}$ is thus given by

$$\mathcal{H} \equiv T_{00} = \frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}(\partial_i\phi)^2 + U(\phi) \quad . \tag{2}$$

With the choice of $U(\phi)$ as described above, the Hamiltonian density is a sum of three positive definite terms. Hence, $\mathcal{H} \geq 0$ and the energy is also positive definite.

The **classical vacuum** is defined to be the classical solution(s) with lowest energy. In order to obtain a solution of lowest energy, one has to minimise the three terms in eqn. (2) separately. The vanishing of the first two terms in the Hamiltonian density implies that ϕ should be a constant (in time and space). The value(s) of this constant is given by the minima of the potential $U(\phi)$. Clearly such a configuration not only minimises the energy but also gives a solution of the Euler-Lagrange equation[1]. There are two distinct possibilities which follow

1. $U(\phi)$ has a unique minimum which implies that the classical vacuum is unique. For e.g., $U(\phi) = \frac{1}{2}m^2\phi^2$ has a minimum energy $E = 0$ for $\phi = 0$ in all space.

[1] For time-independent configurations, one has $\mathcal{L} = -\mathcal{H}$ implying that the extrema of both functions coincide for this sub-class of configurations.

2. $U(\phi)$ has several degenerate minima, say at $\phi = a_1, a_2, \ldots$. This implies that there is a degeneracy in the classical vacuum. As the minimum value of $U(\phi)$ is zero, the classical vacuum has $E = 0$ which occurs when $\phi(x,t) = a_i$, where a_i is any one of the minima of $U(\phi)$.
   For example, $U(\phi) = \frac{\lambda}{2}(\phi^2 - a^2)^2$ has two minima at $\phi = \pm a$ both of which have energy $E = 0$.

In addition to the classical vaccuum, there is another class of solutions to the Euler-Lagrange equation which are of some interest. These are solutions which have **finite energy**. We shall also assume that the solution is **time-independent** even though this is not essential to obtain finite energy. In order to have finite energy, the spatial integral over the Hamiltonian density should not diverge. This implies that at spatial infinity, ϕ must necessarily tend to one of the zeros of $U(\phi)$.

It turns out that for scalar fields, the conditions of finite-energy and time-independence leads to interesting solutions for the case of one time and one spatial dimension, x. We shall now restrict ourselves to $1 + 1$ dimensions. Going back to the two possibilities for $U(\phi)$ discussed above, we obtain the following

1. When $U(\phi)$ has a unique minimum the only finite energy solution is the $E = 0$ solution which is the classical vacuum.

2. When $U(\phi)$ has many zeros, then there exists several possibilities. At $x = \pm\infty$, $U(\phi)$ can now tend to different possible values in order to obtain a finite energy. Thus, if the potential has two zeros, such as for $U(\phi) = \frac{\lambda}{2}(\phi^2 - a^2)^2$, then there are four distinct possibilities. Of these four, two of them are the classical vacua given by $\phi = \pm a$. The other two are given by $\phi \to \pm a$ as $x \to \pm\infty$ and $\phi \to \pm a$ as $x \to \mp\infty$ with energy $0 < E < \infty$. We shall now exhibit such solutions.

1 The kink soliton

Consider the Lagrangian density in 1+1 dimensions with the potential $U(\phi)$

$$U(\phi) = \frac{\lambda}{2}(\phi^2 - a^2)^2 \quad . \tag{3}$$

Let us choose x as the spatial coordinate. We will look for a solution that *interpolates* between the two classical solutions, $\phi(x) = \pm a$. In other words, we look for a solution that has the property

$$\lim_{x\to-\infty} \phi(x) \to -a \quad \text{and} \quad \lim_{x\to+\infty} \phi(x) \to +a \ . \tag{4}$$

Recall that if we are interested in finite-energy solutions, it is necessary (but not sufficient) for the field to go to its classical vacuum at spatial infinity. This is clearly achieved by our choice. Explicit, analytic solutions to the E-L equations subject to such spatial boundary conditions are rare. However, in this example, there is a solution called the *kink soliton* given by

$$\phi_K(x) = a \tanh \mu x \ , \tag{5}$$

with $\mu = +a\sqrt{\lambda}$. Note that one doesn't even have the freedom of multiplying the solution by a constant as that changes the value of the field at spatial infinity and hence affects the finite-energy condition. Further, the Euler-Lagrange equations are non-linear in the field and hence, again, multiplying any solution by a constant does not give rise to another solution.

Exercise: Show that the kink soliton solves the equations of motion *only* when $\mu^2 = \lambda a^2$.

The Hamiltonian density is positive definite since it is the sum of squares

$$\mathcal{H} = \frac{1}{2}\big(\partial_t\phi\big)^2 + \frac{1}{2}\big(\partial_x\phi\big)^2 + \frac{\lambda}{2}(\phi^2 - a^2)^2 \ .$$

Further the momentum density (given by T^{01}) is

$$\mathcal{P} = -\partial_t\phi\partial_x\phi \ .$$

Since our solution is time-independent, it is easy to see that

$$\mathcal{H}(x,t) = \lambda a^4 \ \mathrm{sech}^4\mu x \quad \text{and} \quad \mathcal{P}(x,t) = 0 \ .$$

Integrating the densities over all space, we get

$$E = \int_{-\infty}^{\infty} dx \ \mathcal{H}(x) = \frac{4}{3}\mu a^2 \quad \text{and} \quad P = \int_{-\infty}^{\infty} dx \ \mathcal{P}(x) = 0 \ . \tag{6}$$

A plot of $\phi(x)$ and the energy density $\mathcal{H}(x)$ for the kink soliton is shown below (when $a = \lambda = 1$). Note that the energy density is a maximum at $x = 0$ and most of the total energy is concentrated in a region of width $4/\mu$ centred at $x = 0$. Further, it falls to zero exponentially for large $|x|$, $\mathcal{H}(x) \sim \lambda a^4 \exp(-4\mu|x|)$. Thus, for practical purposes, it looks like an object of size $4/\mu$ at rest with mass $4\mu a^2/3$.

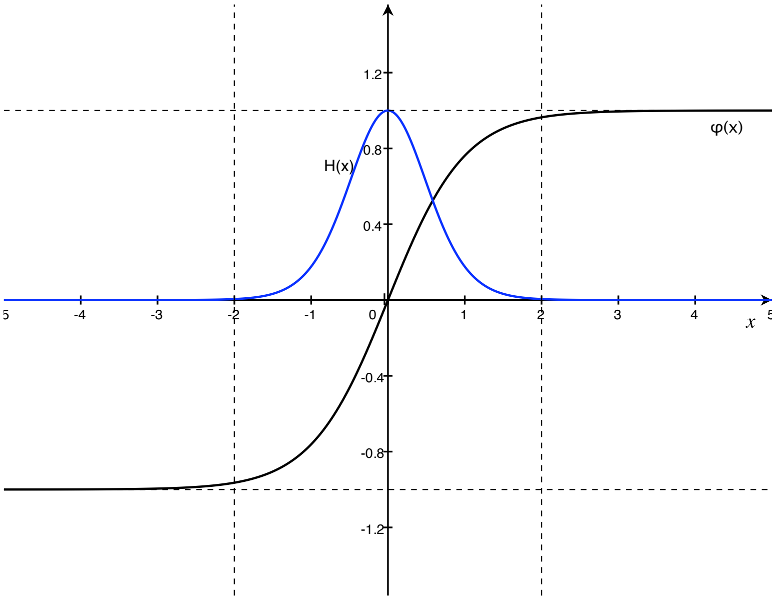

Figure 1: The kink solution and its energy density when $\lambda = a = 1$. Note that 99.8% of the energy lies between in the region $[-2, 2]$.

2 Symmetries

The action is invariant under the 1+1 dimensional Poincaré group consisting of time and spatial translations as well as Lorentz boosts. Further, it is invariant under parity, $x \to -x$. The kink soliton given above clearly is **not** invariant under any of these symmetries. What does this mean? We will now consider the action of these symmetries on the kink solution and generate new solutions. When a solution does not preserve a symmetry of its action, we say that the symmetry is **broken**. Coleman argues that this nomenclature is misleading and prefers to call it as **hidden symmetry**.

Parity: Consider the action of parity. One has $\phi(-x) = -\phi(x)$ – this remains a solution. This new solution goes to $\mp a$ as $x \to \pm\infty$. This is called an anti-kink and is considered as a *new* solution to the equation of motion. Further, the anti-kink has the same energy and momentum as the kink solution.

Time Translation: As the solution is time-independent, it is clearly invariant under time translation. Thus time translation is a symmetry of the solution as well.

Inversion: There is a discrete symmetry which we will call inversion that takes $\phi(x) \to -\phi(x)$. This symmetry is an internal one as it does not act on the coordinates of space-time. It maps the kink to the anti-kink like parity. It is easy to see that inversion is a symmetry of both the vacuum solutions, i.e., $\phi(x) = \pm a$.

Spatial Translation: Next, consider the effect of spatial translation. This acts by shifting the location of the maximum value of the energy density. This leads to a family of time-independent solutions

$$\phi_K(x) = a \tanh \sqrt{\lambda} a(x - x_0) \ , \tag{7}$$

that are parametrised by the constant x_0. All solutions correspond to energy $E = 4\mu a^2/3$ and momentum $P = 0$.

Lorentz boost: Finally, a Lorentz boost of the above solution gives rise to a new *time-dependent* solution of the Euler-Lagrange equation:

$$\phi_{\text{new}}(x, t) = \phi\big[\gamma(x - vt)\big] \ ,$$

where $\gamma \equiv 1/\sqrt{1 - v^2}$. The student may verify directly that this is true. For this solution $\mathcal{P} \neq 0$. A direct calculation shows that

$$E = \frac{4}{3}\mu a^2 \cosh\theta \quad \text{and} \quad P = \frac{4}{3}\mu a^2 \sinh\theta \ ,$$

where $\tanh\theta = v$ as usual. Thus, $(E^2 - P^2)$ is the same for both solutions. Thus, the solution behaves like a particle of rest mass $m = \frac{4}{3}\mu a^2$ since its energy density is concentrated in a region of size about $4/\mu$ as can be seen from the figure!

Unlike the kink solution, the vacuum solution is invariant under the above symmetries and thus new solutions are not generated by their action. However, there is another discrete symmetry, $\phi(x) \to -\phi(x)$ under which the two vacuum solutions, $\phi(x) = \pm a$, are interchanged.

The most general kink solution is thus parametrised by two continuous parameters, x_0 (the location of the kink) and v (the velocity of the kink):

$$\boxed{\phi_K\big(x, t \bigm| x_0, v\big) = a \tanh \sqrt{\lambda}\ a\ \gamma(x - x_0 - vt) \ .} \tag{8}$$

This fact that the degrees of freedom of a kink is the same as that of a particle with one degree of freedom lends credence to our claim that a soliton behaves like a classical one-dimensional particle. Properties of this particle can be established by quantizing these degrees of freedom! We will see another example, the sine-Gordon theory, kinks behave like particles which on quantization behave as fermions.

A more general situation

Let G denote the group of symmetries, both continuous and discrete, of a given action and let $\Phi(x)$ collectively denote all fields that appear in the action. Consider a solution, $\Phi_0(x)$, of the Euler-Lagrange equations of motion. Let $g \in G$ be an arbitrary group element and let ${}^g\Phi_0(x) = g \cdot \Phi_0(x)$ denote the transformation of the solution by the action of g. As it is a symmetry of the theory, ${}^g\Phi_0(x)$ continues to be a solution to the Euler-Lagrange equations of motion for the theory. There are two possibilities for ${}^g\Phi_0(x)$.

1. ${}^g\Phi_0(x) = \Phi_0(x)$: We then say that g is a symmetry of the solution as its action leaves the solution unchanged.

2. ${}^g\Phi_0(x) \neq \Phi_0(x)$: We say the symmetry g is not a symmetry of the solution and gives rise to a new solution. Such symmetries are sometimes called *broken* or, as Coleman prefers, *hidden* symmetries.

It is easy to see that the set of symmetries of a solution that arise from G will form a sub-group, H, of G. It is entirely possible that the symmetries of a given solution might be larger than H but they may not all arise as a symmetry of the action. We are restricting our considerations to only those symmetries that are part of G. It is then easy to see that the hidden symmetries belong to the coset G/H since, for any $h \in H$ one has

$$ {}^g\Phi_0(x) := g \cdot \Phi_0(x) = g \cdot h \cdot \Phi_0(x) = {}^{gh}\Phi_0(x) \ . \qquad (9)$$

In other words, every element of the coset, G/H, gives rise to a new solution. Note that H is determined by the solution that one is considering.

Exercise: Take G to be the $1+1$-dimensional Poincaré group and work out H for the vacuum and kink soliton of the real scalar field that we just considered.

A numerical exercise: We will look for an initial configuration that consists of a kink and an anti-kink widely separated but moving towards each other. For large separations, the energy of the system is about two times the energy of a (moving) kink. At some time $t = -T$, one has

$$\phi_{K+A}(x,t) = \phi_K\big(x,t \,\big|\, -x_0, v\big) - \phi_K\big(x,t \,\big|\, x_0, -v\big) - a \ . \qquad (10)$$

(Why did we subtract a in the above equation?) By numerically integrating the Euler-Lagrange equation, we wish to solve for the field configurations at times $t > -T$. From a numerical viewpoint, it is better to rewrite the Euler-Lagrange equation as a first-order (Hamiltonian) system:

$$\partial_t \begin{pmatrix} \phi(x,t) \\ \pi(x,t) \end{pmatrix} = \begin{pmatrix} \pi(x,t) \\ \partial_x^2 \phi(x,t) - \frac{\partial U(\phi)}{\partial \phi} \end{pmatrix} , \tag{11}$$

subject to the initial conditions at $t = -T$:

$$\phi(x,-T) = \phi_{K+A}(x,-T) \quad \text{and} \quad \pi(x,-T) = \partial_t \phi_{K+A}(x,t)\Big|_{t=-T} .$$

What should the solution look like? We expect the kink and anti-kink to 'collide'. Then, they should bounce off each other – they can't go through each other as that would modify the boundary condition at spatial infinity. The numerical integration should be carried out using the fourth-order Runge-Kutta (RK4) algorithm. Create a movie (animated gif) showing the time-evolution of the energy density as a function of time.

3 Derrick's Theorem

The existence of a kink soliton in $1+1$ dimensions might make one think that we can find such solutions in $d+1$ dimensions with $d > 1$. There is a *no-go theorem* due to Derrick which says that there are **no** time-independent finite-energy solutions (other than the classical vacuum) when $d > 1$ in theories involving *only* scalar fields.

We shall now provide a proof of Derrick's theorem. Let us assume that there exists a time-independent finite-energy solution, $\phi_F(x)$ to the Euler-Lagrange equations. Or equivalently, $\phi_F(x)$ *minimizes* the energy subject to suitable boundary conditions. Let the energy of the solution be

$$E = K_F + V_F , \tag{12}$$

where K_F is the contribution from the $|\nabla \phi_F|^2$ term and V_F is the contribution from the $U(\phi_F)$ term in the energy density. Note that both the constants, K_F and V_F are positive definite.

Consider a one-parameter family of configurations,

$$\phi(x|\alpha) \equiv \phi_F(\alpha x) ,$$

where α is a positive real parameter. A simple calculation shows that the energy of such a configuration is

$$E(\alpha) = \alpha^{2-d}\, K_F + \alpha^{-d}\, V_F . \tag{13}$$

If $\phi_F(x)$ is indeed a (finite energy) solution to the equations of motion, then it must minimise the energy. In other words, the function, $E(\alpha)$ should be such that

$$dE(\alpha)/d\alpha\Big|_{\alpha=1} = 0 \text{ and } d^2E(\alpha)/d\alpha^2\Big|_{\alpha=1} > 0 \quad . \tag{14}$$

Let us check if this is possible. One has

$$\frac{dE(\alpha)}{d\alpha} = (2-d)\ \alpha^{1-d}\ K_F - d\ \alpha^{-d-1}\ V_F\ . \tag{15}$$

Let us check whether this can vanish at $\alpha = 1$.

d = 1 $dE(\alpha)/d\alpha = 0$ at $\alpha = 1$ implies that we need $K_F - V_F = 0$. This is clearly possible. Verify that this occurs for the kink soliton. The condition $K_F = V_F$ is achieved if one requires something stronger

$$\frac{d\phi(x)}{dx} = \pm\sqrt{2U(\phi)}$$

at all points x. Convince yourself that solutions to the above equation are indeed solutions to the Euler-Lagrange equations as well. Does the kink soliton satisfy the stronger condition? If yes, for what sign?

d = 2 $dE(\alpha)/d\alpha = 0$ at $\alpha = 1$ implies that we need $V_F = 0$. This can only happen for the classical vacuum configuration.

d > 2 $dE(\alpha)/d\alpha$ is now the sum of two negative terms and hence can only vanish for the classical vacuum configuration.

This completes our proof of Derrick's theorem. As with most no-go theorems, there are interesting ways around it as we shall see later on in the course.

Suggested reading: The chapter titled "Classical Lumps and their quantum descendants" in Sidney Coleman's book *Aspects of Symmetry* published by the Cambridge University Press.