National Programme on Technology Enhanced Learning*

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Physics – Classical Field Theory **Final Examination**

http://sgovindarajan.wikidot.com/cftontheweb

Duration: 3 hours . **Max. marks: 50**

This paper contains 5 questions in 4 numbered pages. All questions carry equal marks. You may refer to your own ***handwritten*** *class notes. All quantities have their usual meaning and we shall work in natural units where $\hbar = c = 1$ unless stated otherwise.*

1. Indicate whether the statements in quotes are true or false by circling **T** or **F**.

(a) "The Euler-Lagrange equations of motion for a non-abelian gauge field in Yang-Mills theory is linear in the gauge fields."

T **F**

(b) "The Lorentz group in $3+1$ dimensions is of rank two. "

T **F**

(c) "The dimensionality of the fundamental representation is always the same as the number of generators of the group."

T **F**

(d) "The adjoint representation of $SU(N)$ is real."

T **F**

(e) "The electric field generated by a charged particle at rest can be described by a four-vector potential A^μ such that $\mathbf{A} \neq 0, A^0 = 0$."

T **F**

(f) The action for a complex scalar field is given by

$$S = \int d^4x \left[\eta^{\mu\nu}\, \partial_\mu\phi\, \partial_\nu\phi^* - \mu^2|\phi|^2 - \lambda|\phi|^4\right]\,;\ \lambda < 0\ .$$

"We can have spontaneous symmetry breaking when $\mu^2 < 0$."

T **F**

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(g) "The analog of the Dirac quantization condition for two dyons of charges (e_1, g_1) and (e_2, g_2) is $(e_1 g_2 + e_2 g_1) \in \frac{1}{2}\mathbb{Z}$."

T **F**

(h) A gauge field is said to be a pure gauge if the corresponding field strength vanishes identically.
"$A_\mu = \partial_\mu \alpha$ is pure gauge for an abelian gauge field."

T **F**

(i) "$A^a_\mu = \partial_\mu \alpha^a$ is pure gauge for a non-Abelian gauge field."

T **F**

(j) "The Green function associated with the Feynman propagator does not vanish outside the light cone violating causality."

T **F**

2. Answer the following questions in brief.

(a) In the $SU(5)$ GUT model, an $SU(5)$ local gauge symmetry is broken to $SU(3)_c \times SU(2) \times U(1)_Y$ by the Higgs mechanism. Estimate the number of massive vector bosons in this model.

(b) "*It is possible to break* $SU(2) \to U(1)$ *with scalar fields transforming in the fundamental representation of* $SU(2)$." Do you agree with this statement? Justify your answer in a line or two.

(c) Let Φ_α $(\alpha = 1, 2, 3)$ be three complex scalars transforming in the adjoint representation of $SU(N)$ – it is useful to think of each Φ_α as a traceless $N \times N$ matrix. Let them have a Lagrangian (with the standard $SU(N)$ invariant kinetic term) and a potential $U(\Phi)$ given by

$$U(\Phi, \Phi^*) = \sum_\alpha \left| \frac{\partial W}{\partial \Phi_\alpha} \right|^2 ,$$

where $W(\Phi) = \mathrm{Tr}\,(q\Phi_1\Phi_2\Phi_3 - q^*\Phi_1\Phi_3\Phi_2)$ with q a complex number. Obtain the conditions under which the minimum $U = 0$ is attained.

(d) Four flavours of quarks, say u, d, s and c, transform in the 4 of $SU(4)_f$. How many mesons can be formed from these quarks? How many of these mesons will carry strangeness $S = 1$?

(e) The existence of vortex solutions with finite energy in the abelian Higgs model does **not** violate Derrick's theorem. Explain the reason in a couple of sentences.

3. Let T_a denote the generators of the $su(N)$ Lie algebra in the fundamental representation. Recall that this implies that they are $N \times N$ traceless hermitian matrices. Further assume that they are chosen such that

$$\mathrm{Tr}(T_a T_b) = 2\delta_{ab} \quad , \quad [T_a, T_b] = i f_{ab}{}^c\, T_c \ ,$$

where $f_{abc} \equiv f_{ab}{}^e \delta_{ec}$ are the totally antisymmetric structure constants. One can show that the product of two generators is of the form

$$T_a T_b = \alpha\ \delta_{ab} + d_{ab}{}^c\ T_c + \beta\ f_{ab}{}^c\ T_c \ ,$$

where α and β are constants and $d_{abc} = d_{bac}$ is defined by the above expansion.

(a) Determine the constants α and β.

(b) Show that d_{abc} is totally symmetric in its three indices by obtaining an expression for d_{abc} in terms of the generators of the Lie algebra.

(c) Obtain the condition(s) on d^{abc} under which the cubic Casimir defined by

$$\mathcal{C} \equiv d^{abc}\ T_a T_b T_c \ ,$$

commutes with all the generators of the $su(N)$ Lie algebra i.e., $[\mathcal{C}, T_a] = 0\ \forall\ a$.

(d) Does d_{abc} exist for the $su(2)$ Lie algebra? (Recall that we can choose $T_a = \sigma_a$.)

4. The Born-Infeld action for an abelian gauge field is given by

$$S_{BI} = -T \int d^4x\ \left[\sqrt{-\det\left(\eta_{\mu\nu} + \alpha F_{\mu\nu}\right)} - 1\right] ,$$

where T and α' are two constants. In the above expression, the determinant is taken by treating the combination $(\eta + \alpha F)$ as a 4×4 matrix.

(a) Determine the scaling dimensions of the two constants, T and α assuming that the field strength $F_{\mu\nu}$ has its usual scaling dimension.

(b) Show that

$$-\det\left(\eta_{\mu\nu} + \alpha F_{\mu\nu}\right) = 1 + \frac{\alpha^2}{2} F_{\mu\nu}F^{\mu\nu} - \frac{\alpha^4}{16}(F_{\mu\nu}\widetilde{F}^{\mu\nu})^2 \ .$$

where $\widetilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\sigma\rho}$.
Hint: Let $A = (a_{mn})$ be a 4×4 matrix. Then,

$$\det A = \tfrac{1}{4!}\epsilon^{m_1\cdots m_4}\ \epsilon^{n_1\cdots n_4}\ a_{m_1 n_1}\cdots a_{m_4 n_4} \ .$$

(c) By studying the leading terms for small and large values of the field strengths (or equivalently, the constant α), show that the action interpolates between the Maxwell action and a purely topological one.

(d) Obtain the equations of motion for the above Lagrangian.

5. We have seen one way of getting around Derrick's theorem in the lectures. Another way to do it is to add higher derivative terms to the Lagrangian density. In this regard, consider the Skyrme-Faddeev model in 3 + 1 dimensions where the field is a unit vector, $\mathbf{n}(x)$, in some auxiliary (not physical) space, $\mathbf{n} = (n_1, n_2, n_3)$ with $\mathbf{n}^2 = n_1^2 + n_2^2 + n_3^2 = 1$):

$$S = \int d^4x \left[\frac{1}{2e^2}(\partial_\mu \mathbf{n})^2 - \frac{1}{4g^2}\big(\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n}\big)^2 + \lambda\big(\mathbf{n}^2 - 1\big) \right] .$$

Note that while the action has up to four derivatives, the time derivatives appear only quadratically. Further, we have added a Lagrange multiplier λ whose equation of motion imposes the constraint $\mathbf{n}^2 = 1$. Thus, we can treat $\mathbf{n}$ as a unconstrained field in what follows.

(a) Compute the momenta, π_a, conjugate to n_a and hence compute the energy density $\mathcal{H}$ as the Legendre transform of the Lagrangian density.

(b) Show that for time-independent configurations, the energy is given by the positive definite quantity

$$E = E_2 + E_4 = \int d^3x \left[\frac{1}{2e^2}(\partial_i \mathbf{n})^2 + \frac{1}{4g^2}\big(\partial_i \mathbf{n} \times \partial_j \mathbf{n}\big)^2 \right] ,$$

after imposing the constraint $\mathbf{n}^2 = 1$. Note that E_2 refers to the term quadratic in derivatives and E_4 refers to the term quartic in derivatives. It is easy to see that $\mathbf{n}$ must equal some constant vector $\mathbf{n}_0$ at spatial infinity for finite energy solutions.

(c) Show that under a scaling of spatial coordinates $x_i \to \kappa x_i$, the two terms scale as $E_2 + E_4 \to \kappa^\alpha E_2 + \kappa^\beta E_4$ for some constants, α and β, to be determined by you.

(d) Repeating the argument that we used to prove Derrick's theorem for the existence of *stable finite-energy time-independent* configurations, show that there is indeed a minimum value of E (taken to occur at $\kappa = 1$) for which $E_2 = cE_4$ for some c (again to be determined by you). This shows that this model admits stable time-independent finite energy solutions.