National Programme on Technology Enhanced Learning*

http://nptel.iitm.ac.in/courses/115106058/

Physics – Classical Field Theory **Problem Set 1**

http://sgovindarajan.wikidot.com/cftontheweb

(To be solved before viewing lecture 2)

1. A particle of mass m and charge q moves under the influence of an electromagnetic field. The electric and magnetic fields (not necessarily time-independent) can be written in terms of the scalar potential $\phi(\mathbf{x}, t)$ and vector potential $\mathbf{A}(\mathbf{x}, t)$ as follows:

$$\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t \ , \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A} \ .$$

Show that the Euler-Lagrange equations for the Lagrangian given below reproduces the Lorentz force law:

$$L = \frac{1}{2} m \ \dot{\mathbf{x}}^2 + q \ \dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}, t) - q \ \phi(\mathbf{x}, t) \ .$$

This is an example of a Lagrangian that is **not** of the form $T - V$ as it involves a velocity-dependent conservative force. How does the Lagrangian change if we work with another equivalent choice of potentials, $\phi'(\mathbf{x}, t)$ and $\mathbf{A}'(\mathbf{x}, t)$, that give the same electromagnetic field?

2. The Hamiltonian for a system with one degree of freedom has the form

$$H = \frac{p^2}{2a} - bqpe^{-\alpha t} + \frac{ba}{2} q^2 e^{-\alpha t}(\alpha + be^{-\alpha t}) + \frac{kq^2}{2} \quad ,$$

where a, b, α, and k are constants.

   (a) Find a Lagrangian corresponding to this Hamiltonian.

   (b) Find an equivalent Lagrangian that is not explicitly dependent on time.

   (c) What is the Hamiltonian corresponding to the second Lagrangian, and what is its relationship between the two Hamiltonians?

3. The point of suspension of a simple pendulum of length l and mass m is constrained to move on a parabola $z = ax^2$ in the vertical $x - z$ plane. Derive a Hamiltonian governing the motion of the pendulum and its point of suspension. Obtain the Hamilton's equations of motion.

4. A free particle of unit mass moving in one dimension with coordinate q. We are given that the particle was $q = 1$ at $t = 0$ and at $q = 5$ at time $t = 1$. The action for this particle is

$$S[q(t)] = \frac{1}{2}\int_0^1 dt\,(\dot{q})^2\ .$$

Consider the following four possible trajectories of the particles:

$$\begin{aligned} q_A(t) &= 2\sin(2\pi t) + 4t + 1\ , \\ q_B(t) &= -3t^2 + 7t + 1\ , \\ q_C(t) &= 4t + 1\ , \\ q_D(t) &= -2\cos(\pi t) + 3\ . \end{aligned}$$

Of the four paths, $q_C(t)$ is the solution to the equations of motion – this is usually called the *classical solution*. Compute the action for all four trajectories and verify that the minimum occurs for $q_C(t)$.

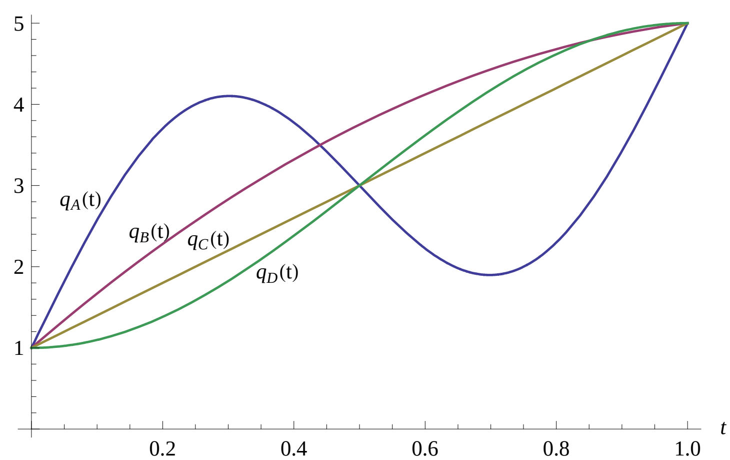

Figure 1: A plot of the four trajectories. Note that all paths begin at $q = 1$ and end at $q = 5$.

5. Consider the action for a particle of unit mass in a harmonic potential

$$S[q(t)] = \frac{1}{2}\int_0^1 dt\,\left[(\dot{q})^2 - \pi^2(q-3)^2\right]\ .$$

Evaluate the action for the four trajectories that were considered in the previous problem – the initial and final conditions being the same as well. Show that the minimum value occurs for the solution $q_D(t)$ and verify that it is indeed the classical solution.