

National Programme on Technology Enhanced Learning*

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Physics – Classical Field Theory

Problem Set 11

<http://sgovindarajan.wikidot.com/cftontheweb>

(To be solved while viewing lectures 30/31/32)

Consider the Lagrangian density in 3+1 dimensions which is invariant under local $SO(3)$ transformations[†]

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu a}F_{\mu\nu}^a + \frac{1}{2}\pi^{\mu a}\pi_{\mu}^a + \frac{1}{2}\mu^2\phi^a\phi^a - \frac{1}{4}\lambda(\phi^a\phi^a)^2 \quad ,$$

where $F_{\mu\nu}^a = (\partial_{\mu}A_{\nu}^a - \partial_{\nu}A_{\mu}^a + e\epsilon^{abc}A_{\mu}^bA_{\nu}^c)$; and $\pi_{\mu}^a = \partial_{\mu}\phi^a + e\epsilon^{abc}A_{\mu}^b\phi^c$. ($a, b, c = 1, 2, 3$)

1. Obtain the Euler-Lagrange equations of motion.
2. Consider the following time-independent ansatz

$$\begin{aligned} A_0^a &= 0 \quad , \\ A_i^a &= \epsilon_{aij}x_j[1 - K(r)]/er^2 \quad , \\ \phi^a &= x_a H(r)/er^2 \quad , \end{aligned}$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Substitute the ansatz in the equations of motion and show that the equations of motion reduce to

$$\begin{aligned} r^2 K'' &= K(K^2 - 1) + KH^2 \quad , \\ r^2 H'' &= 2HK^2 + \frac{\lambda}{e^2}(H^3 - C^2 r^2 H) \quad , \end{aligned}$$

where $C = \mu e/\sqrt{\lambda}$ and the prime denotes differentiation w.r.t. r . Obtain the condition(s) on H and K such that the Hamiltonian (as well as the Lagrangian) is finite.

3. In the limit where $\lambda, \mu \rightarrow 0$ keeping the ratio $C = \mu e/\sqrt{\lambda}$ constant, show that $K = Cr/\sinh(Cr)$ and $H = Cr \coth(Cr) - 1$ solve the equations of motion. This limit is usually referred to as the *Prasad-Sommerfield* limit. We will call this solution as the Prasad-Sommerfield (PS) solution.
4. For large r , the PS solution reduces to the classical vacuum where the $SO(3)$ gauge symmetry is broken to $U(1)$. The field strength of this $U(1)$ field is given by

$$\mathcal{F}_{\mu\nu} = \partial_{\mu}(\hat{\phi}^a A_{\nu}^a) - \partial_{\nu}(\hat{\phi}^a A_{\mu}^a) - (1/e)\epsilon^{abc}\hat{\phi}^a\partial_{\mu}\hat{\phi}^b\partial_{\nu}\hat{\phi}^c \quad ,$$

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[†]We are deliberately using the notation of Prasad and Sommerfield to encourage the student to see their paper!

where $\hat{\phi}^a \equiv \phi^a / (\phi^b \phi^b)^{1/2}$. This definition has been chosen based on the observation that if after a gauge transformation $\phi^a = \delta^{a3}$ everywhere within a region, then $\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu^3 - \partial_\nu A_\mu^3$ in that region. Verify this observation. What is that gauge transformation for large r for the PS solution? Obtain the electric and magnetic charge of the soliton from this field strength for the PS solution? Compare the solution with that of the Dirac monopole in the large r region.

5. Obtain the mass of the soliton by first deriving the Hamiltonian density and then integrating it over all space in the usual way.

Remark: The remarkable thing about the Prasad-Sommerfield solution is that it is a solution to a set of first-order equations in addition to being a solution of the usual second-order Euler-Lagrange equations of motion (in the PS limit discussed above). These first-order equations are due to Bogomolny (or Bogomolnyi) and are called *Bogomolnyi equations*. Such solutions are thus called BPS solutions. It turns out that these equations (and hence the solutions) naturally arise in supersymmetric field theories. Today, when one says BPS equations, BPS bounds etc. it usually refers to a field theory in a supersymmetric setting. There is some discussion in the last lecture of this course in the context of instantons (discussed in the next problem set) and string theory.

References:

1. P.A.M. Dirac, Quantised Singularities in the Electromagnetic Field, Proc. Roy. Soc. **A133**, 60 (1931).
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3. G. 't Hooft, *Magnetic monopoles in unified gauge theories*, Nucl. Phys. **B79**, 276 (1974).
4. A.M. Polyakov, *Particle spectrum in quantum field theory*, JETP Lett. **20**, 194 (1974)
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6. E. B. Bogomolny, *Stability of Classical Solutions*, Sov. J. Nucl. Phys. **24** (1976) 449 [Yad. Fiz. **24** (1976) 861].