National Programme on Technology Enhanced Learning[*]

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Physics – Classical Field Theory **Problem Set 12**

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(To be solved while viewing lectures 34–37)

In four-dimensional Euclidean space with metric $\delta_{\mu\nu}$, consider the Lagrangian density associated with $SU(2)$ gauge fields ($\mathcal{A}_\mu = \sum_{a=1}^3 A_\mu^a T_a$ where T_a's are in the fundamental representation of $SU(2)$)[†]

$$\mathcal{L} = -\frac{1}{2e^2}\, tr_F(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}) = -\frac{1}{4e^2}\sum_{a=1}^{3} F_{\mu\nu}^a F^{\mu\nu a} \quad ,$$

where $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - i[\mathcal{A}_\mu, \mathcal{A}_\nu]$ is the field strength. Using $\mathcal{F}_{\mu\nu} = F_{\mu\nu}^a T_a$, one can show that $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \epsilon_{abc} A_\mu^b A_\nu^c$.

1. Obtain the equation of motion for A_μ^a. For subsequent use, rewrite in terms of the matrix valued objects, $\mathcal{A}_\mu$ and $\mathcal{F}_{\mu\nu}$.

2. Consider the following *one-instanton* solution to the equations of motion
$$\mathcal{A}_a = \frac{-\epsilon_{abc}\sigma_b x_c + \sigma_a x_4}{x^2+\lambda^2} \quad , \quad \mathcal{A}_4 = \frac{-\sigma_a x_a}{x^2+\lambda^2}$$
The parameter λ is the "size" of the *instanton*. Verify the following expression for the field strength by computing $\mathcal{F}_{a4}$ from the above expression for the gauge field.
$$\mathcal{F}_{a4} = \frac{-2\lambda^2\sigma_a}{(x^2+\lambda^2)^2} \quad , \quad \mathcal{F}_{ab} = \frac{-2\lambda^2\epsilon_{abc}\sigma_c}{(x^2+\lambda^2)^2} \quad .$$

3. Verify that the above is indeed a solution to the equations of motion. It is sufficient to verify it for one component.

4. Show that the above solution leads to a finite action and evaluate its value. Solutions which lead to finite action (as opposed to energy) are called *instanton* solutions.

References

1. A.A. Belavin, A.M. Polyakov, A.S. Schwartz, Yu.S.Tyupkin, *Pseudoparticle solutions of the Yang-Mills equations* . Phys.Lett. **B59** (1975) 8587.

2. G. 't Hooft, *Computation of the Quantum Effects Due to a Four-Dimensional Pseudoparticle*, Phys. Rev. D **14** (1976) 3432 [Erratum-ibid. D **18** (1978) 2199].

3. M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld and Y. .I. Manin, *Construction of Instantons*, Phys. Lett. A **65** (1978) 185.

The third reference is mentioned in the last lecture happens to be 1.5 pages long and is called the ADHM construction. It nontrivially generalises the above solution to arbitrary Lie groups.

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[†]The explicit realisation of the T_a's are not important. However, you may need the relations: $[T_a, T_b] = i\epsilon_{abc}T_c$ and $tr_F(T_aT_b) = \delta_{ab}/2$. Indices are raised and lowered using δ_{ab}.