National Programme on Technology Enhanced Learning*

http://nptel.iitm.ac.in/courses/115106058/

Physics – Classical Field Theory **Problem Set 2**

http://sgovindarajan.wikidot.com/cftontheweb

(To be attempted after viewing lectures 2 and 3)

1. An arbitrary spatial rotation can be written as a rotation by an angle θ about an axis (specified by a unit vector $\hat{n}$). This is only true in three dimensions. Why?. Let us call the corresponding rotation matrix $R(\hat{n}, \theta)$. Obtain an explicit expression for this 3×3 matrix.
   Recommended reading: V. Balakrishnan, *How is a vector rotated?*, Resonance, Vol. **4**, No. 10, pp. 61-68 (1999). Also available at the URL: http://www.physics.iitm.ac.in/~labs/dynamical/pedagogy/

2. An arbitrary Galilean transformation is given by ten parameters (the rotation matrix R, the boosts $\mathbf{u}$, the spatial translations $\mathbf{a}$ and time translations t_0):

$$x'^i = R^i{}_j \; x^j + u^i \; t + a^i$$
$$t' = t + t_0$$

   (a) Obtain the inverse of the above transformation.

   (b) Consider the composition of a Galilean transformation with parameters (R, $\mathbf{u}$, $\mathbf{a}$, t_0) with another Galilean transformation with parameters (R', $\mathbf{u}'$, $\mathbf{a}'$, t'_0). Show that you obtain another Galilean transformation by explicitly determining its parameters.

3. Recall that $\partial^\mu = (\partial_0, -\nabla)$ and $A^\mu = (\phi, \mathbf{A})$ in Gaussian units. Define the electromagnetic field strength as follows:

$$F^{\mu\nu} = -F^{\nu\mu} := (\partial^\mu A^\nu - \partial^\nu A^\nu) \; .$$

   (a) Show that $F^{i0} = E^i$ and $F^{ij} = -\epsilon^{ijk} B_k$ where E^i and B_i are the electric and magnetic fields. This shows that the electric and magnetic field combine to form a *second-rank antisymmetric tensor* (this will be defined in lecture 4).

   (b) Show that the Maxwell's equations with sources in Gaussian units can be written as

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$$

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where $J^\mu = (c\rho, \mathbf{J})$ and $\partial_\mu = (\partial_0, \nabla)$. Using the explicit definition of the field strength in terms of the vector potential, it is easy to verify the following identity, called the *Bianchi identity*:

$$\partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu} + \partial^\rho F^{\mu\nu} = 0 \ .$$

Not surprisingly, these reduce to the Maxwell equations not containing source terms when $\mu \neq \nu \neq \rho \neq \mu$ as one can explicitly verify.

4. A particle of mass m and charge q is at rest in a region is space with constant and uniform magnetic field $\mathbf{B} = B\ \hat{e}_z$.

   (a) Obtain a vector potential A^μ corresponding to this magnetic field.

   (b) By using the expression for a Lorentz boost given in lecture 3, work out the vector potential A'^μ in an inertial frame moving with constant velocity $\mathbf{u} = u\ \hat{e}_x$ with respect to the particle.

   (c) Hence, compute the electric and magnetic fields in the moving frame from A'^μ.

   (d) Show that the Lorentz force on the particle is zero in the moving frame. This is consistent with the fact that there is no force in the frame where the particle is stationary.

Remark on the scalar and vector potential: In lecture 2, the vector potential $\mathbf{A}(\mathbf{x})$ was introduced as a 'solution' to $\nabla \cdot \mathbf{B} = 0$. When can we write $\mathbf{B} = \nabla \times \mathbf{A}$? It is useful to recall the Helmholtz decomposition in vector calculus. It states that a smooth vector field in $\mathbb{R}^3$, $\mathbf{B}(\mathbf{x})$, can be decomposed as

$$\mathbf{B}(\mathbf{x}) = -\nabla G(\mathbf{x}) + \nabla \times \mathbf{A}(\mathbf{x}) \ .$$

For the magnetic field, the above decomposition implies that $\nabla^2 G(\mathbf{x}) = 0$ everywhere. Does this always imply that we can choose $G = 0$? The answer depends on the domain in which the magnetic field is defined. When the domain is $\mathbb{R}^3$, the only solution is when G is a constant. If we impose the condition that G vanishes at spatial infinity, it follows that $G = 0$. This will no longer be true if we delete one point, say the origin, from $\mathbb{R}^3$. Then, $G = a/|\mathbf{x}|$ for some constant a will satisfy $\nabla^2 G(\mathbf{x}) = 0$. Such a situation will indeed appear in the latter part of this course when we discuss Dirac monopoles. A similar issue arises with the scalar potential and we request the student to study this example along with a review of the Helmholtz theorem.

One has to remember that there are some assumptions about the nature of the spatial domain as well as boundary conditions in order to define the scalar and vector potentials.