National Programme on Technology Enhanced Learning*

http://nptel.iitm.ac.in/courses/115106058/

Physics – Classical Field Theory **Problem Set 3**

http://sgovindarajan.wikidot.com/cftontheweb

(To be solved while/after viewing lectures 4 and 5)

1. Let $O^\mu{}_\nu$ $(\mu, \nu = 0, 1, 2, 3)$ denote a $O(1,3)$ matrix satisfying

$$(O^\mu{}_\rho)^T\ \eta_{\mu\nu}\ O^\nu{}_\sigma = O^T{}_\rho{}^\mu\ \eta_{\mu\nu}\ O^\nu{}_\sigma = \eta_{\rho\sigma}\ ,$$

with the metric $\eta_{\mu\nu} = \text{Diag}(1, -1, -1, -1)$.

   (a) Show that $\det(O) = \pm 1$ and $|O^0{}_0| \geq 1$.

   (b) Let $SO(1,3)^+$ denote the subgroup of $O(1,3)$ matrices with det $= 1$ and sign$(O^0{}_0)$ being positive. Verify that the product of two $SO(1,3)^+$ matrices is indeed another $SO(1,3)^+$ matrix.
   Remark: We shall refer to the group $SO(1,3)^+$ as the **Lorentz** group in this course – this is sometimes referred to as the orthochronous Lorentz group since it preserves the arrow of time. Inadvertantly, this was not explicitly mentioned in the lectures!

   (c) Let $\lambda^{\mu\nu}$ be an antisymmetric matrix. From this construct a new matrix $\lambda^\mu{}_\nu \equiv \lambda^{\mu\rho}\eta_{\rho\nu}$. Show that $\exp(\lambda^\mu{}_\nu)$ is a $SO(1,3)^+$ matrix.

   (d) Time reversal is denoted by the matrix $T = \text{Diag}(-1, 1, 1, 1)$ and parity is denoted by the matrix $P = \text{Diag}(1, -1, -1, -1)$. Verify that $T \in O(1,3)^-$ and $P \in O(1,3)^+$ where the superscript denotes the sign of $O^0{}_0$. Show that an arbitrary $O(1,3)$ matrix can be converted to an $SO(1,3)^+$ matrix by multiplying it by a suitable combination of P and/or T.

   (e) An arbitrary Lorentz boost with rapidity, ϕ, along a direction $\hat{n}$ can be represented by a 4×4 matrix $B(\hat{n}, \phi)$. Show that $B(\hat{n}, \phi)$ is given by

$$\begin{pmatrix} \cosh\phi & n_1 \sinh\phi & n_2 \sinh\phi & n_3 \sinh\phi \\ n_1 \sinh\phi & 1 - n_1^2(1-\cosh\phi) & -n_1 n_2(1-\cosh\phi) & -n_1 n_3(1-\cosh\phi) \\ n_2 \sinh\phi & -n_1 n_2(1-\cosh\phi) & 1 - n_2^2(1-\cosh\phi) & -n_2 n_3(1-\cosh\phi) \\ n_3 \sinh\phi & -n_1 n_3(1-\cosh\phi) & -n_2 n_3(1-\cosh\phi) & 1 - n_3^2(1-\cosh\phi) \end{pmatrix}.$$

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In the limit of small rapidity, show that one recovers a Galilean boost along the direction $\hat{n}$ and speed $u = c\phi$.

2. Let $x'^{\mu} = O^{\mu}{}_{\nu} x^{\nu} + a^{\mu}$ represent an arbitrary transformation under the Poincaré group. Here the matrix O parametrizes Lorentz transformations and a^{μ} parametrizes translations in spacetime. Consider two such transformations in succession with parameters (O_1, a_1) and (O_2, a_2) and obtain the composition rule. Verify that all the axioms of a group are satisfied.

3. Let Ω be the $2n \times 2n$ antisymmetric matrix, $\left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$, where I_n is the $n \times n$ identity matrix. A symplectic $2n \times 2n$ matrix S (with real entries) satisfies the symplectic condition
$$S \cdot \Omega \cdot S^T = \Omega \ .$$
   (a) Verify that the set of all such matrices form a group, called the symplectic group $Sp(2n, \mathbb{R})$.
   (b) Show that $\det S = +1$.
   (c) Writing $S = \exp(\lambda K)$, first obtain the condition(s) on the matrix K such that the symplectic condition holds to first order in λ.
   (d) Next, writing $K = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in terms of $n \times n$ matrices A, B, C and D, work out the condition(s) on these four matrices. Note that $K^T = \left(\begin{smallmatrix} A^T & C^T \\ B^T & D^T \end{smallmatrix}\right)$ – a common mistake is to forget to transpose the B/C matrices! It is best verified by explicitly writing out the matrices for $n = 2$.

4. Obtain the condition under which the 2×2 matrix U given below is an $SU(2)$ matrix.
$$U = \alpha_0 \ I + i\alpha_j \ \sigma^j \quad ,$$
where σ^j are the Pauli sigma matrices and α_0, α_j are real parameters. What is the geometric interpretation of this condition? What happens if you choose α_0, α_j to be complex? You may need the following identity involving Pauli matrices:
$$\sigma^j \sigma^k = \delta^{jk} \ I + i\epsilon^{jk}{}_l \ \sigma^l$$

5. Since the Pauli sigma matrices form a basis for traceless 2×2 hermitian matrices, an arbitrary $SU(2)$ matrix (in the two dimensional representation) can also be written as
$$U^{(2)}(\hat{n}, \theta) = \exp\left(i\frac{\theta}{2}\hat{n} \cdot \vec{\sigma}\right) \quad ,$$
where $|\hat{n}| = 1$. Solve for α_0, α_i in terms of θ and $\hat{n}$. Hence show that $U(\hat{n}, 2\pi) = -I$.